Math 54 – HW 2 solutions

1.3.1 $\mathbf{u}+\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$; $\mathbf{u}-2\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

1.3.5 $$\begin{array}{lcccl} 3x_1 & + & 5x_2 & = & 2 \\ -2x_1 & & & = & -3 \\ 8x_1 & + & -9x_2 & = & 8 \end{array}$$

1.3.9 $x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \mathbf{0}$

1.3.11 We row reduce the augmented matrix corresponding to the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a_2} + x_3\mathbf{a_3} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the last column of this augmented matrix is not a pivot column, the system is consistent. So $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. (Note: the solution in the book is wrong)

1.3.14 The corresponding augmented matrix is the same as in the previous problem, so $\mathbf{b}$ is a linear combination of the columns of A.

1.3.23

(a) False; $\begin{bmatrix} -4 & 3 \end{bmatrix}$ is a 1×2 matrix, not a (column) vector.

(b) False; the plane determined by these vectors is all of $\mathbb{R}^2$.

(c) True; $\frac{1}{2}\mathbf{v}_1 = \frac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2$.

(d) True; the definition of equality for column vectors gives that the vector equation is equal to the system of equations with the given augmented matric.

(e) False; for example Span$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is just the line through the origin and the point $(1, 0)$.

1.3.24

(a) False; for example Span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is the $x - y$ plane.

(b) I'm not sure about this one. Maybe it is supposed to be false: only *ordered* lists of 5 real numbers are vectors in $\mathbb{R}^5$. To me, lists are the same thing as ordered lists; so I think it is true.

(c) True; $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ iff $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ iff $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution.

(d) True; by our rules about vector arithmetic, $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u} + (-\mathbf{v} + \mathbf{v}) = \mathbf{u} + \mathbf{0} = \mathbf{u}$.

(e) False; the weights can be any real numbers.

1.4.11 The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ 0 & 2 & -6 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

So $\mathbf{x} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}$ is the only solution.

1.4.13 Asking if $\mathbf{u}$ is in the plan spanned by the columnd of A amounts to asking if $A\mathbf{x} = \mathbf{u}$ is consistent. We write and row reduce the corresponding augmented matrix.

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 4 \\ 3 & -5 & 0 \\ -1 & 3 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & -8 & -12 \\ 0 & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last column is not a pivot column, so the system is consistent. Hence, $\mathbf{u}$ is in the plane spanned by the columns of A.

1.4.15 We write and row reduce the augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 3 & -1 & b_1 \\ -9 & 3 & b_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & -1 & b_1 \\ -3 & 1 & \frac{b_2}{3} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & -1 & b_1 \\ 0 & 0 & \frac{b_2}{3} + b_1 \end{bmatrix}.$$

This matrix equation has a solution iff the last column of out final matrix is not a pivot column iff $\frac{b_2}{3} + b_1 = 0$ iff $b_2 = -3b_1$. So, for example, for $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A\mathbf{x} = \mathbf{b}$ has no solutions.

1.4.24

(a) True; see Theorem 3.

(b) True; the corresponding vector equation witnesses that $\mathbf{b}$ is a linear combination of the columns of A.

(c) True; if $\mathbf{b} = c_1\mathbf{a}_1 + \ldots + c_n\mathbf{a}_n$, then letting A be the matrix with columns the $\mathbf{a}_i$ (in order) and $\mathbf{c}$ the column vector with entries c_i (in order), $\mathbf{b} = A\mathbf{c}$.

(d) False; if the coefficient matrix has a pivot position in every row, then the corresponding matrix equation is *consistent* (as the last column of the corresponding *augmented* matrix cannot have a pivot position).

(e) True; see Theorem 3.

(f) False; if the columns *do* span $\mathbb{R}^m$, then the equation is consistent for any $\mathbf{b}$. (For a counter example to the claim, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $A\mathbf{x} = \mathbf{b}$ is inconsistent.)

1.4.25 We read of the values of c_i from the entries of the column vector with which we multiply the matrix (i.e. the column vector on the LHS of the equation):

$c_1 = -3,\ c_2 = -1, c_3 = 2$.

1.4.29 $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1.4.30 $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (e.g. $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not the span of the columns).

1.4.31 A 3×2 matrix A can have at most 2 pivot positions, so A cannot have a pivot position in every row. So Theorem 5 (d) fails, hence Theorem 5 (a) is false: $A\mathbf{x} = \mathbf{b}$ does not have a solution for every $\mathbf{b}$. By the same argument, if A is an $m \times n$ matrix with $m < n$, then $A\mathbf{x} = \mathbf{b}$ does not have a solution for every $\mathbf{b}$.

1.4.34 We'll show that $\mathbf{u}_1 + \mathbf{u}_2$ is a solution to $A\mathbf{x} = \mathbf{w}$. By the properties of matrix-vector multiplication, $A(\mathbf{u}_1 + \mathbf{u}_2) = A\mathbf{u}_1 + A\mathbf{u}_2 = \mathbf{v_1} + \mathbf{v}_2 = \mathbf{w}$.

1.5.1 We write and row reduce the corresponding augmented matrix.

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that only the first two columns are pivot columns, so x_3 is free. Hence, the equation has a nontrivial solution.

1.5.5 Again, we row reduce the corresponding augmented matrix.

$$\begin{bmatrix} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $\mathbf{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

1.5.9 We row reduce the given matrix.

$$\begin{bmatrix} 3 & -6 & -6 \\ -2 & 4 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $\mathbf{x} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

1.5.24

(a) False; $\mathbf{0}$ is always a solution to a homogeneous equation.

(b) False; if $\mathbf{x}$ is a nontrivial solution to a homogeneous equation, then *some entry* in $\mathbf{x}$ is nonzero, i.e. $\mathbf{x} \neq \mathbf{0}$.

(c) True; see the discussion on the middle of page 47.

(d) True; if $\mathbf{0}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{0} = A\mathbf{0} = \mathbf{b}$.

(e) True; see Theorem 6 and the discussion following it.

1.5.25

(a) By the rules for matrix vector multiplication and choice of $\mathbf{p}, \mathbf{v}_h$,

$A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$. So $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ is a solution to $A\mathbf{x} = \mathbf{b}$.

(b) Similarly, $A(\mathbf{w} - \mathbf{p}) = A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$. So $\mathbf{w} - \mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{0}$ and $\mathbf{w} = \mathbf{p} + (\mathbf{w} - \mathbf{p})$.

1.5.38 $A(c\mathbf{w}) = c(A\mathbf{w}) = c\mathbf{0} = \mathbf{0}$.

1.5.39 $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

$A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\mathbf{0} + d\mathbf{0} = \mathbf{0}$.

1.7.1 We row reduce the matrix with the given vectors as columns.

$\begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 7 & 9 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 7 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. This matrix has a pivot in every column, so $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So, the vectors are linearly independent.

1.7.7 The matrix has four columns and three rows, so by Theorem 8, the set of column vectors is linearly dependent.

1.7.9 We row reduce the matrix with columns the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\begin{bmatrix} 1 & 3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & h-10 \end{bmatrix}.$$

No matter the value of h, this matrix will not have pivot columns in every column. So $\mathbf{v}_3 \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Also, $\mathbf{v}_2 = -3\mathbf{v}_1$, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

1.7.11 We row reduce the matrix with columns the given vectors.
$\begin{bmatrix} 2 & 4 & -2 \\ -2 & -6 & 2 \\ 4 & 7 & h \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & h+4 \end{bmatrix}$. The vectors are linearly dependent iff the last column is not a pivot column, i.e. iff $h = -4$.

1.7.22

(a) True; if $\mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{v}\}$, then $\mathbf{w}$ is a linear combination of $\mathbf{u}, \mathbf{v}$ so $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

(b) True; if three vectors in $\mathbb{R}^3$ lie in the same plane, then one of the vectors lies in the plane determined by the other two, i.e. one vector is a linear combination of the others.

(c) False; for example $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly dependent.

(d) False; for any n, $\{\mathbf{0}\}$ is linearly independent (where $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$).

1.7.31 Label the columns (in order) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Then, by the observation from the problem statement, $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, i.e. $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. So, reading off the coefficients, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a nontrivial solution.

1.7.32 Label the columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Then $\mathbf{v}_3 = \mathbf{v}_1 - 3\mathbf{v}_2$, i.e. $\mathbf{v}_1 - 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. So $\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$ is a nontrivial solution.

1.7.33 True; $\mathbf{v}_4$ is a linear combination of the other vectors.

1.7.34 False; take $\mathbf{v}_1 = \mathbf{0}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $v_2 \neq c\mathbf{v}_2$ for any scalar c, but $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

1.7.37 True; since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, one of the vectors is a linear combination of the other two (e.g. $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$). But then this vector is a linear combination of the the other two and $\mathbf{v}_4$ (e.g. $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4$).

1.7.38 True; otherwise $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_3\}$ would be linearly dependent, so by 1.7.37, $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ would be linearly dependent too, contradicting the assumption that it is linearly independent.