MATH 54 Homework 1 Solutions

1.1

1.1.1

$$\left(\begin{array}{cc|c} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right)$$

We conclude $x_1 = -8, x_2 = 3$

1.1.3

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 2 & 4 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

We conclude $x_1 = 2, x_2 = 1$.

1.1.5

We have the maximal number of pivots, so all that remains is to remove the numbers above each pivot. We should work from the bottom up, so the next two row operations are to subtract 4 times the 3rd row from the 2nd, and to add 3 times the 3rd row from the 1st. This would make the 3rd pivot have 0's above it.

1.1.7

$$\left(\begin{array}{ccc|c} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

The third equation corresponds to the linear equation $0x_1 + 0x_2 + 0x_3 = 1$, which has no solutions for x_1, x_2, x_3. We conclude our system has no solutions since the solutions need to satisfy all the linear equations at once, but no points satisfy the third equation.

This system is inconsistent.

1.1.11

$$\left(\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

The last equation corresponds to $0 = -2$, which is not satisfied for any assignment of x_1, x_2, x_3.

1.1.15

$$\left(\begin{array}{cccc|c} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ -1 & 6 & 1 & 5 & 3 \\ 0 & -1 & 5 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & -1 & 5 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 5 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & -8 \end{array}\right)$$

Because we have a pivot in every row our linear system is consistent.

1.1.20

$$\left(\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right)$$

If $-8-2h \neq 0$ then we have a pivot in both rows so the system is consistent. On the other hand if $-8-2h = 0$, then our bottom equation is $0 = 16$, so our system is inconsistent.

We conclude our system is consistent exactly when $h \neq -4$

1.1.23

a

True Row swaps are the inverses of themselves, while adding a multiple of a row to another can be inverted by subtracting, and multiplying a row by a nonzero scalar can be inverted by multiplying by the inverse of the scalar.

b

False A 5×6 matrix has 5 rows and 6 columns

c

False When solving a linear system we are looking for the set of all tuples $(s_1, \dots, s_n)$ that satisfy all of the linear equations at once, not just a single solution.

d

True We often want to know when a linear system is consistent (existence), and how big the solution space is (uniqueness)

1.1.24

a

False Two matrices are row equivalent if we can use row operations to get from one to another. For example the following two matrices have the same number of rows (and columns!) but are not row equivalent

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This can be seen since doing any row operations to the all zero matrix does not change it.

b

True Row reduction is an algorithm that uses operations which do not change the solution set of an augmented matrix to take any matrix into a simpler form in which the solution set is easily recoverable.

c

False Two matrices are equivalent if I can get from one to the other using row operations. Since row operations fix the solution set, we have equivalent matrices have equivalent solution sets.

d

True Consistent linear system indicated existence of a solution. Inconsistent linear systems are ones with no solutions.

1.1.28

One can apply any row operations to the following augmented matrix to obtain (infinitely) many matrices with that solution set:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Conversely, given any augmented matrix with this solution set, if one row reduces we should end up with the previous matrix.

Some examples are:

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

1.2

1.2.1

a

This is in reduced row echelon form because all the pivots are 1 and there are no entries above the pivots.

b

This is in reduced row echelon form because all the pivots rae 1 and there are no entries above the pivots.

c

It is neither since there is a zero row above a nonzero row

d

It is in echelon form, but not reduced row echelon form since there are nonzero elements above the pivots and the pivots are not 1

1.2.5

There are two possible configurations with 1 pivot

$$\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$$

There is one possible configuration with 2 pivots

$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$$

1.2.7

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

If we name our variables x, y, z. Then x, z are our basic variables, and y is a free variable. We get our solutions are

$$x = -5 - 3y$$

$$z = 3$$

where y is any real number.

1.2.11

$$\left(\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & -2 & 4 & 0 \end{array}\right)$$

Then x is a basic variable and y, z are free variables. So we get the solution set is

$$x = \frac{2y - 4z}{3}$$

for all y, z any real number

1.2.15

a

We can eliminate a zero row, so we have a pivot in every row so it is consistent. There is a free variable, so the solutions are not unique.

b

There is a pivot in every row, so the matrix is consistent. The first column is a free variable, so the solution is not unique.

1.2.23

The augmented matrix in row reduced echelon form must look like:

$$\left(\begin{array}{cccc|c} \blacksquare & 0 & 0 & 0 & * \\ 0 & \blacksquare & 0 & 0 & * \\ 0 & 0 & \blacksquare & 0 & * \\ 0 & 0 & 0 & \blacksquare & * \end{array}\right)$$

Then we have a pivot in every row so we have existence of a solution, and no free variables, so we have uniqueness of a solution.

1.2.26

If a 3×5 coefficient matrix has 3 pivots, then since we have at most one pivot in every row, we must have at least 3 rows with pivots in them, which means every row has a pivot in it.

Notice that a linear system is consistent if **coefficient** matrix has a pivot in every row.

The corresponding statement for the **augmented** *matrix is that the last column contains no pivots*

1.2.30

$$1x_1 + 2x_2 + 3x_3 = 4$$

$$0x_1 + 0x_2 + 0x_3 = 1$$

The second equation is never satisfied so there are no solutions two both equations simultaneously.