Math 54 – HW 35 solutions

Section 9.4

20.

(a) The Wronskian is the determinant of $\begin{bmatrix} \cos t & \sin t & \cos t \\ 0 & \cos t & \sin t \\ 0 & \cos t & \cos t \end{bmatrix}$. To compute this expand by the first column. Hence $W(t) = \cos t(\cos t * \cos t - \sin t * \cos t) = \cos^2 t(\cos t - \sin t)$

(b) Note that $W(0) = \cos^2 0(\cos 0 - \sin 0) = 1$. As the wronskian is non-zero at some point the vectors are linearly independent.

(c) No. We will prove by contradiction. Suppose there is a matix $A(t)$ such that $x_1'(t) = A(t)x_1(t)$. Note that $x_1 = \begin{bmatrix} -\sin t \\ 0 \\ 0 \end{bmatrix}$. Thus at $t = \pi/2$ we have $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = A(0) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ This is impossible as the zero vector remains zero under multiplication by any matrix.

31. The determinant $= t^2 * (2|t|) - t|t| * (2t) = 2t^2|t| - 2t^2|2| = 0$.

Now suppose the vector functions are linearly dependent. Hence there exists non zero constants a, b such that $a \begin{bmatrix} t^2 \\ 2t \end{bmatrix} = b \begin{bmatrix} t|t| \\ 2|t| \end{bmatrix}$ $\forall t$. thus at $t = 1$ we get that $a \begin{bmatrix} 1 \\ 2 \end{bmatrix} = b \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow$ $a = b$.

However at $t = -1$ we get that $a \begin{bmatrix} 1 \\ -2 \end{bmatrix} = b \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow a = -b$.

Then combining $a = b$ and $a = -b$ we get that $a = b = 0$. Thus this contradicts the assumption that a and b are non zero. Thus the two vectors are linearly independent.

Section 9.6

3. $\chi(A) = (x - 1)((x - 1)^2 + 1)$. Thus the eigenvalues are 1, $1 + i$ and $1 - i$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 - 2i \\ 1 \\ i \end{bmatrix}$, $\begin{bmatrix} 1 + 2i \\ 1 \\ -i \end{bmatrix}$. Thus the general solution is

$$\begin{bmatrix} e^t & (1 - 2i)e^{1+i} & (1 + 2i)e^{1-i} \\ 0 & e^{1+i} & e^{1-i} \\ 0 & ie^{1+2i} & -ie^{1+2i} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

5. $\chi(A) = (x+1)^2 + 16$. Thus the eigenvalues are $-1+4i$ and $-1-4i$. The corresponding eigenvectors are $\begin{bmatrix} 2 \\ -4i \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4i \end{bmatrix}$. Thus the fundamental matrix is $\begin{bmatrix} 2e^{-1+4i} & 2e^{-1-4i} \\ -4ie^{-1+4i} & 4ie^{-1-4i} \end{bmatrix}$

13.

(a) $\chi(A) = x^2+4x+5$. Thus the eigenvalues are $-2+i$ and $-2-i$. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1-i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1+i \end{bmatrix}$. Now using the formula that $x = X(t)X(0)^{-1}x_0$, we get $x = (1/2i)X(t)\begin{bmatrix} -1+i & -1 \\ i+1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{(-2+i)t} & e^{(-2-i)t} \\ (-1-i)e^{(-2+i)t} & (-1+i)e^{(-2+i)t} \end{bmatrix}\begin{bmatrix} -1/2-i/2 \\ -1/2+i/2 \end{bmatrix}$

(c) Same thing gives us $x = X(t)X(-2\pi)^{-1}x_{-2\pi} = (e^{-4\pi}/2i)X(t)\begin{bmatrix} -1+i & -1 \\ i+1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= e^{-4\pi}\begin{bmatrix} e^{(-2+i)t} & e^{(-2-i)t} \\ (-1-i)e^{(-2+i)t} & (-1+i)e^{(-2+i)t} \end{bmatrix}\begin{bmatrix} i/2 \\ -i/2 \end{bmatrix}$

20. Using the normal form equation given in (11) of system (10), and plugging in the values of m_1, m_2, k_1, k_2 we get $y' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} y.$

Now the characteristic polynomial of the above matrix is $r^{+}4r^2+3=0$. Thus the eigenvalues are $\sqrt{3}i$,$-\sqrt{3}i$,i,$-i$.

The corresponding eigenvectors are $\begin{bmatrix} 1 \\ \sqrt{3}i \\ -1 \\ -\sqrt{3}i \end{bmatrix}$, $\begin{bmatrix} 1 \\ -\sqrt{3}i \\ 1 \\ \sqrt{3}i \end{bmatrix}$, $\begin{bmatrix} 1 \\ i \\ 1 \\ i \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -i \\ 1 \\ -i \end{bmatrix}$.

Now using this to solve for the initial conditions by $y = X(t)X(0)^{-1}y_0$ gives us $x_1 = \cos t - \cos\sqrt{3}t$ and $x_2 = \cos t + \cos\sqrt{3}t$

Section 9.7

2. The eigenvalues are 1 and 2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

To find a particular solution assume that $\begin{bmatrix} at+b \\ ct+d \end{bmatrix}$ is a solution. Then plugging this into the equation $x' = Ax + f$, we get $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} at+b \\ ct+d \end{bmatrix} + \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix} = \begin{bmatrix} (a+c-1)t+(b+d-1) \\ (4a+c-4)t+(4b+d-2) \end{bmatrix}$. Thus we get the equations $a+c-1=0$, $b+d-1=a$, $4a+c-4=0$ and $4b+d-2=c$. Solving the 1st and 3rd equations first gives us $a=1$ and $c=0$. Plugging them into the 2nd and 4th equation gives us $b=0$ and $d=2$. Thus the general solution is $\begin{bmatrix} e^t & e^{2t} \\ 2e^t & 2e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} t \\ 2 \end{bmatrix}$

8. Since $f(t)$ contains atmost polynomials of degree 2, a particular solution must be of the form $x_p = t^2a + tb + c$ where $a, b, c \in \mathbb{R}^2$

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Section 10.2

1. The auxilaaty equation is $r^2 - 1 = 0$. Thus the general solution is $y(x) = c_1e^x + c_2e^{-x}$. Putting in the initial value conditions we get $c_1e^0 + c_2e^0 = 0 \Rightarrow c_1 + c_2 = 0$ and $c_1e^1 + c_2e^{-1} = -4$. The first equation gives $c_1 = -c_2$. Plugging this into the second one we get $c_1(e - 1/e) = -4 \Rightarrow c_1 = \frac{-4e}{e^2-1}$ and $c_2 = \frac{4e}{e^2-1}$.

3. Again the auxillary equation is $r^2 + 4 = 0$. Thus the general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x$. Putting in the initial conditions we get $y(0) = c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$. And $y'(\pi) = -2c_1 sin2\pi + 2c_2 cos2\pi = 0 \Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0$. Therefore the only solution to the initial value problem is the zero function.

5. First we find the general solution to the associated homogeneous equation i.e. $y'' - y = 0$. The general solution to that is $c_1e^x + c_2e^{-x}$. Now we try to find a particular solution to the equation. So let us assume $y_p(x) = a + bx \Rightarrow y'' = 0$. So the equation $y'' - y = 1 - 2x$ becomes $-a - bx = 1 - 2x$. Therefore $2x - 1$ is a solution to the differential equation. Hence the general solution is $y(x) = 2x - 1 + c_1e^x + c_2e^{-x}$. Putting in the initial values we get, $y(0) = -1 + c_1 + c_2 = 0 \Rightarrow c_1 + c_2 = 1$ and $y(1) = 2 - 1 + c_1e + c_2e^{-1} = 1 + e \Rightarrow c_1e + c_2e^{-1} = e$. The two equations solve to give $c_1 = 1; c_2 = 0$. Therefore the solution to the initial value problem is $2x - 1 + e^x$

9. Suppose $\lambda < 0$. Then $y(x) = c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x}$. Now putting in the initial values we get $0 = y(0) = c_1e^{\sqrt{\lambda}0} + c_2e^{-\sqrt{\lambda}0} = c_1 + c_2 \Rightarrow c_1 = -c_2$, thus c_1 and c_2 are of opposite sign. And $0 = y'(\pi) = c_1\sqrt{\lambda}e^{\sqrt{\lambda}\pi} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}\pi} \Rightarrow c_1\sqrt{\lambda}e^{\sqrt{\lambda}\pi} = c_2\sqrt{\lambda}e^{-\sqrt{\lambda}\pi} \Rightarrow c_1 = e^{-2\sqrt{\lambda}\pi}c_2$, thus c_1 and c_2 are of same sign. They cant both be of opposite sign and same signs unless $c_1 = c_2 = 0$. Thus negative values of λ yields no nontrivial solution.

   Suppose $\lambda = 0$. Then $y(x) = c_1 + c_2x$ is the general solution. Again putting in the initial values we get $0 = y(0) = c_1$ and $0 = y'(\pi) = c_2$. Thus we got $c_1 = c_2 = 0$ and there is no non trivial solution.

   Suppose $\lambda > 0$. Then the general solution has got to be $c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$. Putting in the initial conditions gives us $0 = y(0) = c_1 \cos \sqrt{\lambda}0 + c_2 \sin \sqrt{\lambda}0 = c_1$. Thus $y(x) = c_2 \sin \sqrt{\lambda}x$ And $0 = y'(\pi) = c_2\sqrt{\lambda} \cos \sqrt{\lambda}\pi$ Thus for c_2 to be not 0, $\cos \sqrt{\lambda}\pi$ must be 0. Since the only places $\cos t = 0$ is when $t = n\pi + \pi/2$ where n is an integer, we must have $\sqrt{\lambda}\pi = n\pi + \pi/2 \Rightarrow \sqrt{\lambda} = n + 1/2 \Rightarrow \lambda = (n + 1/2)^2$. Those are the only values of λ where the equation has nontrivial solution.

12. As in problem (9), there is no non-trivial solution when $\lambda \leq 0$.

    If $\lambda > 0$, the general solution has got to be $c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. Putting in the initial conditions gives us $0 = y(0) = c_1 \cos\sqrt{\lambda}0 + c_2 \sin\sqrt{\lambda}0 = c_1$. Thus $y(x) = c_2 \sin\sqrt{\lambda}x$ And $0 = y'(\pi/2) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}\pi/2$ Thus for c_2 to be not 0, $\cos\sqrt{\lambda}\pi/2$ must be 0. Since the only places $\cos t = 0$ is when $t = n\pi + \pi/2$ where n is an integer, we must have $\sqrt{\lambda}\pi/2 = n\pi + \pi/2 \Rightarrow \sqrt{\lambda} = 2n+1 \Rightarrow \lambda = (2n+1)^2$. Those are the only values of λ where the equation has nontrivial solution.

15. Using formula(15) in the book., we get that $u(x,t) = \sum c_n e^{-\beta(n\pi/L)^2 t} \sin\frac{n\pi x}{L}$ if $f(x) = \sum c_n \sin\frac{n\pi x}{L}$. In the problem, $L = \pi$ and $f(x) = \sin x - 6\sin 4x = \sin\frac{1*\pi*x}{\pi} - 6\sin\frac{4*\pi*x}{\pi} = \sin\frac{1*\pi*x}{L} + (-6)\sin\frac{4*\pi*x}{L}$ [As $L = \pi$].

    Thus $c_1 = 1$ and $c_4 = -6$ and $c_n = 0$ for every other n. Thus the solution is $e^{-3(1*\pi/L)^2 t}\sin\frac{1*\pi*x}{L} - 6e^{-3(4*\pi/L)^2 t}\sin\frac{4*\pi*x}{L} = e^{-3t}\sin x - 6e^{-48t}\sin 4x$.

16. As in previous problem, $f(x) = \sin 3x + 5\sin 7x - 2\sin 13x = \sin\frac{3*\pi*x}{L} + 5\sin\frac{7*\pi*x}{L} + (-2)\sin\frac{13*\pi*x}{L}$ [since $L = \pi$].

    Thus $c_3 = 1$, $c_7 = 5$, $c_{13} = -2$ and $c_n = 0$ for any other n. Thus applying the formula, the solution is $e^{-3(3*\pi/L)^2 t}\sin\frac{3*\pi*x}{L} + 5e^{-3(7*\pi/L)^2 t}\sin\frac{7*\pi*x}{L} - 2e^{-3(13*\pi/L)^2 t}\sin\frac{13*\pi*x}{L} = e^{-27t}\sin 3x + 5e^{-147t}\sin 7x - 2e^{-507t}\sin 13x$