Math 54 – HW 35 solutions

Section 9.4

20.

(a) The Wronskian is the determinant of
\[
\begin{bmatrix}
\cos t & \sin t & \cos t \\
0 & \cos t & \sin t \\
0 & \cos t & \cos t \\
\end{bmatrix}
\]. To compute this expand by the first column. Hence
\[
W(t) = \cos t (\cos t \cos t - \sin t \sin t) = \cos^2 t (\cos t - \sin t)
\]

(b) Note that
\[
W(0) = \cos^2 0 (\cos 0 - \sin 0) = 1.
\]
As the wronskian is non-zero at some point the vectors are linearly independent.

(c) No. We will prove by contradiction. Suppose there is a matrix
\[
A(t) \text{ such that } x_1'(t) = A(t)x_1(t).
\]
Note that
\[
x_1 = \begin{bmatrix}
-\sin t \\
0 \\
0 \\
\end{bmatrix}.
\]
Thus at \( t = \pi/2 \) we have
\[
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} = A(0) \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
This is impossible as the zero vector remains zero under multiplication by any matrix.

31. The determinant is \( t^2 * (2|t|) - t|t| * (2t) = 2t^2|t| - 2t^2|2| = 0 \).

Now suppose the vector functions are linearly dependent. Hence there exists non zero constants \( a, b \) such that
\[
a \begin{bmatrix} t^2 \\ 2t \end{bmatrix} = b \begin{bmatrix} t|t| \\ 2|t| \end{bmatrix} \forall t.
\]
Thus at \( t = 1 \) we get that
\[
a \begin{bmatrix} 1 \\ 2 \end{bmatrix} = b \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow a = b.
\]

However at \( t = -1 \) we get that
\[
a \begin{bmatrix} 1 \\ -2 \end{bmatrix} = b \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow a = -b.
\]

Then combining \( a = b \) and \( a = -b \) we get that \( a = b = 0 \). Thus this contradicts the assumption that \( a \) and \( b \) are non zero. Thus the two vectors are linearly independent.

Section 9.6

3. \( \chi(A) = (x-1)((x-1)^2 + 1) \). Thus the eigenvalues are \( 1, 1+i \) and \( 1-i \). The corresponding eigenvectors are
\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-2i \\ 1 \\ i \end{bmatrix}, \begin{bmatrix} 1+2i \\ 1 \\ -i \end{bmatrix}.
\]
Thus the general solution is
\[
\begin{bmatrix} e^t \quad (1-2i)e^{1+i} \\ e^{1+i} \\ ie^{1+2i} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
\]

5. \( \chi(A) = (x+1)^2 + 16 \). Thus the eigenvalues are \( -1+4i \) and \( -1-4i \). The corresponding eigenvectors are
\[
\begin{bmatrix} 2 \\ -4i \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 4i \end{bmatrix}.
\]
Thus the fundamental matrix is
\[
\begin{bmatrix} 2e^{-1+4i} & 2e^{-1-4i} \\ -4ie^{-1+4i} & 4ie^{-1-4i} \end{bmatrix}
\]

13. \( \chi(A) = x^2 + 4x + 5 \). Thus the eigenvalues are \(-2 + i\) and \(-2 - i\). The corresponding eigenvectors are \([\begin{array} {c} 1 \\ -1 - i \end{array}] \) and \([\begin{array} {c} 1 \\ -1 + i \end{array}] \). Now using the formula that \( x = X(t)X(0)^{-1}x_0 \), we get \( x = (1/2i)X(t) \). The eigenvalues are 1 and 2. The corresponding eigenvectors are \( a = 1 \) and \( d \). Thus the general solution is \( c_1 \cos t + c_2 \sin t \).

(c) Same thing gives us \( x = X(t)X(-2\pi)^{-1}x_{-2\pi} = (e^{-4\pi}/2i)X(t) \). Now using this to solve for the initial conditions by \( y = X(t)X(0)^{-1}y_0 \) gives us \( x_1 = \cos t - \cos \sqrt{3}t \) and \( x_2 = \cos t + \cos \sqrt{3}t \).

Section 9.7

2. The eigenvalues are 1 and 2. The corresponding eigenvectors are \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ -2 \end{bmatrix} \). To find a particular solution assume that \( \begin{bmatrix} at + b \\ ct + d \end{bmatrix} \) is a solution. Then plugging this into the equation \( x' = Ax + f \), we get \( \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} at + b \\ ct + d \end{bmatrix} + \begin{bmatrix} -t - 1 \\ -4t - 2 \end{bmatrix} = \begin{bmatrix} (a + c - 1)t + (b + d - 1) \\ (4a + c - 4)t + (4b + d - 2) \end{bmatrix} \). Thus we get the equations \( a + c - 1 = 0 \), \( b + d - 1 = a \), \( 4a + c - 4 = 0 \) and \( 4b + d - 2 = c \). Solving the 1st and 3rd equations first gives us \( a = 1 \) and \( c = 0 \). Plugging them into the 2nd and 4th equation gives us \( b = 0 \) and \( d = 2 \). Thus the general solution is \( \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} t \\ 2t \end{bmatrix} \).
8. Since \( f(t) \) contains atmost polynomials of degree 2, a particular solution must be of the form \( x_p = t^2a + tb + c \) where \( a, b, c \in \mathbb{R}^2 \)

**Math 54 – HW 37 solutions**

**Section 10.2**

1. The auxiliary equation is \( r^2 - 1 = 0 \). Thus the general solution is \( y(x) = c_1e^x + c_2e^{-x} \). Putting in the initial value conditions we get \( c_1e^0 + c_2e^0 = 0 \Rightarrow c_1 + c_2 = 0 \) and \( c_1e^1 + c_2e^{-1} = -4 \). The first equation gives \( c_1 = -c_2 \). Plugging this into the second one we get \( c_1(e - 1/e) = -4 \Rightarrow c_1 = \frac{-4e}{e^2 - 1} \) and \( c_2 = \frac{4e}{e^2 - 1} \).

3. Again the auxiliary equation is \( r^2 + 4 = 0 \). Thus the general solution is \( y(x) = c_1 \cos 2x + c_2 \sin 2x \). Putting in the initial conditions we get \( y(0) = c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0 \). And \( y'(\pi) = -2c_1 \sin 2\pi + 2c_2 \cos 2\pi = 0 \Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0 \). Therefore the only solution to the initial value problem is the zero function.

5. First we find the general solution to the associated homogeneous equation i.e. \( y'' - y = 0 \). The general solution to that is \( c_1e^x + c_2e^{-x} \). Now we try to find a particular solution to the equation. So let us assume \( y_p(x) = a + bx \Rightarrow y'' = 0 \). So the equation \( y'' - y = 1 - 2x \) becomes \( -a - bx = 1 - 2x \). Therefore \( 2x - 1 \) is a solution to the differential equation. Hence the general solution is \( y(x) = 2x - 1 + c_1e^x + c_2e^{-x} \). Putting in the initial values we get, \( y(0) = -1 + c_1 + c_2 = 0 \Rightarrow c_1 + c_2 = 1 \) and \( y(1) = 2 - 1 + c_1e + c_2e^{-1} = e \Rightarrow c_1e + c_2e^{-1} = e \). The two equations solve to give \( c_1 = 1; c_2 = 0 \). Therefore the solution to the initial value problem is \( 2x - 1 + e^x \).

9. Suppose \( \lambda < 0 \). Then \( y(x) = c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x} \). Now putting in the initial values we get \( 0 = y(0) = c_1e^{\sqrt{\lambda}0} + c_2e^{-\sqrt{\lambda}0} = c_1 + c_2 \Rightarrow c_1 = -c_2 \), thus \( c_1 \) and \( c_2 \) are of opposite sign. And \( 0 = y'(\pi) = c_1\sqrt{\lambda}e^{\sqrt{\lambda} \pi} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda} \pi} \Rightarrow c_1\sqrt{\lambda}e^{\sqrt{\lambda} \pi} = c_2\sqrt{\lambda}e^{-\sqrt{\lambda} \pi} \Rightarrow c_1 = e^{-2\sqrt{\lambda} \pi}c_2 \), thus \( c_1 \) and \( c_2 \) are of same sign. They cant both be of opposite sign and same signs unless \( c_1 = c_2 = 0 \). Thus negative values of \( \lambda \) yields no nontrivial solution.

Suppose \( \lambda = 0 \). Then \( y(x) = c_1 + c_2x \) is the general solution. Again putting in the initial values we get \( 0 = y(0) = c_1 \) and \( 0 = y'(\pi) = c_2 \). Thus we got \( c_1 = c_2 = 0 \) and there is no non trivial solution.

Suppose \( \lambda > 0 \). Then the general solution has got to be \( c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x \). Putting in the initial conditions gives us \( 0 = y(0) = c_1 \cos \sqrt{\lambda}0 + c_2 \sin \sqrt{\lambda}0 = c_1 \) and \( y(x) = c_2 \sin \sqrt{\lambda}x \). And \( 0 = y'(\pi) = c_2\sqrt{\lambda} \cos \sqrt{\lambda} \pi \). Thus for \( c_2 \) to be not 0, \( \cos \sqrt{\lambda} \pi \) must be 0. Since the only places \( \cos t = 0 \) is when \( t = n\pi + \pi/2 \) where \( n \) is an integer, we must have \( \sqrt{\lambda} \pi = n\pi + \pi/2 \Rightarrow \sqrt{\lambda} = n + 1/2 \Rightarrow \lambda = (n + 1/2)^2 \). Those are the only values of \( \lambda \) where the equation has nontrivial solution.
12. As in problem (9), there is no non-trivial solution when \( \lambda \leq 0 \).

If \( \lambda > 0 \), the general solution has got to be \( c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x \). Putting in the initial conditions gives us \( 0 = y(0) = c_1 \cos \sqrt{\lambda}0 + c_2 \sin \sqrt{\lambda}0 = c_1 \). Thus \( y(x) = c_2 \sin \sqrt{\lambda}x \) and \( 0 = y'(\pi/2) = c_2 \sqrt{\lambda} \cos \sqrt{\lambda} \pi/2 \) Thus for \( c_2 \) to be not 0, \( \cos \sqrt{\lambda} \pi/2 \) must be 0. Since the only places \( \cos t = 0 \) is when \( t = n\pi + \pi/2 \) where \( n \) is an integer, we must have \( \sqrt{\lambda} \pi/2 = n\pi + \pi/2 \Rightarrow \sqrt{\lambda} = 2n + 1 \Rightarrow \lambda = (2n + 1)^2 \). Those are the only values of \( \lambda \) where the equation has nontrivial solution.

15. Using formula(15) in the book, we get that
\[
\begin{align*}
  u(x,t) &= \sum c_n e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L} \quad \text{if} \quad f(x) = \sum c_n \sin \frac{n\pi x}{L} \quad \text{[As} \quad L = \pi] \\
  &\quad \text{Thus} \quad c_1 = 1, \quad c_4 = -6 \quad \text{and} \quad c_n = 0 \quad \text{for every other} \quad n. \quad \text{Thus the solution is}
  
  e^{-3(1\pi/L)^2 t} \sin \frac{1\pi x}{L} - 6e^{-3(4\pi/L)^2 t} \sin \frac{4\pi x}{L} = e^{-3t} \sin x - 6e^{-48t} \sin 4x.
\end{align*}
\]

16. As in previous problem, \( f(x) = \sin 3x + 5 \sin 7x - 2 \sin 13x = \sin \frac{3\pi x}{L} + 5 \sin \frac{7\pi x}{L} + (-2) \sin \frac{13\pi x}{L} \quad \text{[since} \quad L = \pi] \).

Thus \( c_3 = 1, \quad c_7 = 5, \quad c_{13} = -2 \quad \text{and} \quad c_n = 0 \quad \text{for any other} \quad n. \quad \text{Thus applying the formula, the solution is}
\[
  e^{-3(3\pi/L)^2 t} \sin \frac{3\pi x}{L} + 5e^{-3(7\pi/L)^2 t} \sin \frac{7\pi x}{L} - 2e^{-3(13\pi/L)^2 t} \sin \frac{13\pi x}{L} = e^{-27t} \sin 3x + 5e^{-147t} \sin 7x - 2e^{-507t} \sin 13x
\]