Math 54 – HW 12 solutions

Section 9.1

9.1.3

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

9.1.7 Let $v = y'$. Assuming $m \neq 0$ this means that

$$v' = y'' = \frac{1}{m}(-by' - ky) = -\frac{b}{m}v - \frac{k}{m}y$$

And therefore we can write the system as

$$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$$

9.1.10 Let $v = y'$. This means that

$$v' = y'' = \frac{-1}{t}y' - \left(1 - \frac{n^2}{t^2}\right) y = \frac{-1}{t}v - \left(1 - \frac{n^2}{t^2}\right) y$$

And therefore we can write the system as

$$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -(1 - n^2/t^2) & -1/t \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$$

Section 9.4

9.4.3

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} t^2 & -1 & -1 \\ 0 & 0 & e^t \\ t & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t \\ 5 \\ -e^t \end{bmatrix}$$

9.4.5 Let $v = y'$. This means that

$$v' = y'' = 3y' + 10y + \sin(t) = 3v + 10y + \sin(t)$$

And therefore we can write the system as

$$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$

9.4.7 Let $x = w'$, $y = x' = w''$ and $z = y' = w'''$. Thus $\frac{d^4w}{dt^4} = \frac{dz}{dt}$. So we have

$$z' = -w + t^2$$

Therefore we can write the system as

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}$$

9.4.20 (a) By expanding along the first column, we can compute the Wronskian:

$$\begin{aligned} W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t) &= \det \begin{bmatrix} \cos t & \sin t & \cos t \\ 0 & \cos t & \sin t \\ 0 & \cos t & \cos t \end{bmatrix} \\ &= \cos(t)(\cos^2(t) - \cos(t)\sin(t)) \\ &= \cos^2(t)(\cos(t) - \sin(t)) \end{aligned}$$

(b) The functions are linearly independent. To see this, we can plug 0 into the Wronskian to get

$$W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](0) = \cos^2(0)(\cos(0) - \sin(0)) = 1.$$

Since the Wronskian is nonzero at some point in $(-\infty, \infty)$, the functions are linearly independent.

(c) Note that if we plug $\pi/2$ into the Wronskian we find

$$W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](\pi/2) = \cos^2(\pi/2)(\cos(\pi/2) - \sin(\pi/2)) = 0.$$

Since the Wronskian is neither always 0 nor always nonzero on the interval $(-\infty, \infty)$ we can conclude that there is no homogeneous linear system for which the functions are solutions.

9.4.21 To check if $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, we can check if the Wronskian is nonzero.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{bmatrix} e^{2t} & -2e^{2t} \\ -2e^{2t} & 4e^{2t} \end{bmatrix} = 4e^{4t} - 4e^{4t} = 0.$$

Since the Wronskian is identically zero, the functions are linearly dependent and thus do not form a fundamental solution set, which would require two linearly independent solutions. By the way, in general the Wronskian of some functions may be zero everywhere even if the functions are linearly independent, but if the functions are all solutions to a linear differential equation (as is the case here) then that will not occur.

9.4.23 To check if $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent, we can check if the Wronskian is nonzero (alternatively we could note that the first coordinates of the three functions are linearly independent and thus the functions themselves must be linearly independent).

$$W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t) = \det \begin{bmatrix} e^{-t} & e^t & e^{3t} \\ 2e^{-t} & 0 & -e^{3t} \\ e^{-t} & e^t & 2e^{3t} \end{bmatrix} = e^{-t}(e^{4t}) - e^t(4e^{2t} + e^{2t}) + e^{3t}(-2) = -3e^{3t}.$$

Since the Wronskian is not zero, the functions are linearly independent and since there are three of them, they form a fundamental solution set.

So a fundamental matrix is

$$\begin{bmatrix} e^{-t} & e^t & e^{3t} \\ 2e^{-t} & 0 & -e^{3t} \\ e^{-t} & e^t & 2e^{3t} \end{bmatrix}$$

and a general solution is

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$$

where c_1, c_2, c_3 are any constants.

9.4.26 For any $t_0 \in (-\infty, \infty)$ we have

$$A\mathbf{x}_1(t_0) = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} e^{3t_0} \\ 0 \\ e^{3t_0} \end{bmatrix} = \begin{bmatrix} 3e^{3t_0} \\ 0 \\ 3e^{3t_0} \end{bmatrix} = \mathbf{x}_1'(t_0)$$

Which shows that $\mathbf{x}_1$ is a solution to the homogeneous system on the interval $(-\infty, \infty)$. Similar calculations show that $\mathbf{x}_2$ and $\mathbf{x}_3$ are also solutions. Alternatively, we can simply note that the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

are eigenvectors of A with corresponding eigenvalues $3, 3$, and -3.

To verify that $\mathbf{x}_p$ is a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, observe that

$$A\mathbf{x}_p + \mathbf{f} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix} + \begin{bmatrix} -9t \\ 0 \\ -18t \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = \mathbf{x}_p'$$

Therefore a general solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ is $\mathbf{x}_p$ plus any function in the span of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$– i.e.

$$\begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix} + a \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + b \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{bmatrix} + c \begin{bmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{bmatrix}$$

where a, b, c are any scalars.

9.4.27 Let $\mathbf{x}$ and $\mathbf{y}$ be any two differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$ (or in the words of the textbook, "$n \times 1$ differentiable vector functions") and let a be a scalar. Then for any $t \in \mathbb{R}$ we have

$$\begin{aligned} L[a\mathbf{x}+\mathbf{y}](t) &= (a\mathbf{x}+\mathbf{y})'(t) + (A(a\mathbf{x}+\mathbf{y}))(t) \\ &= a\mathbf{x}'(t) + \mathbf{y}'(t) + A(a\mathbf{x}(t) + \mathbf{y}(t)) \\ &= a\mathbf{x}'(t) + \mathbf{y}'(t) + a(A\mathbf{x}(t)) + A\mathbf{y}(t) \\ &= aL[\mathbf{x}](t) + L[\mathbf{y}](t) \end{aligned}$$

and thus L is a linear transformation.

9.4.28 First we check that $\mathbf{x}(t)$ satisfies $\mathbf{x}' = A\mathbf{x}$. One way to see this is to note that $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0$ is a linear combination of the columns of $\mathbf{X}(t)$. Since each column of $\mathbf{X}(t)$ is a solution to the differential equation, so is $\mathbf{x}(t)$.

Alternatively, we can directly calculate the derivative of $\mathbf{x}(t)$. First observe that since $\mathbf{X}^{-1}(t_0)\mathbf{x}_0$ is constant, we have

$$\mathbf{x}'(t) = \mathbf{X}'(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0.$$

Note that by $\mathbf{X}'(t)$ we mean the matrix whose entries are the derivatives of the corresponding entries in $\mathbf{X}(t)$. Also, it is necessary but straightforward to check that the derivative of $\mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}_0$ can be calculated as we did above. Next observe that since the columns of $\mathbf{X}(t)$ are solutions to $\mathbf{x}' = A\mathbf{x}$ we have

$$\mathbf{X}'(t) = A\mathbf{X}(t).$$

And thus we have

$$\mathbf{x}'(t) = (A\mathbf{X}(t))\mathbf{X}^{-1}(t_0)\mathbf{x}_0 = A\mathbf{x}(t).$$

Next we will check that $\mathbf{x}(t)$ satisfies the required initial value:

$$\mathbf{x}(t_0) = \mathbf{X}(t_0)\mathbf{X}^{-1}(t_0)\mathbf{x}_0 = \mathbf{x}_0.$$

9.4.37 (a) First note that each of e^t, e^{2t} and e^{3t} are solutions to the given differential equation, which we can check by simply plugging each one into the equation in (11) and verifying that the result is 0. As an example

$$(e^{2t})''' - 6(e^{2t})'' + 11(e^{2t})' - 6e^{2t} = 8e^{2t} - 24e^{2t} + 22e^{2t} - 6e^{2t} = 0.$$

Furthermore, since the Wronskian calculated in part (b) is nonzero, the functions are linearly independent and thus form a fundamental solution set.

(b)

$$\begin{aligned} W[e^t, e^{2t}, e^{3t}](t) &= \det \begin{bmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{bmatrix} \\ &= e^t(18e^{5t} - 12e^{5t}) - e^{2t}(9e^{4t} - 3e^{4t}) + e^{3t}(4e^{3t} - 2e^{3t}) \\ &= 2e^{6t} \end{aligned}$$

(c) Let $x_1 = y$, $x_2 = y' = x_1'$ and $x_3 = y'' = x_2'$. Thus $x_3' = y''' = 6y'' - 11y' + 6y = 6x_3 - 11x_2 + 6x_1$. Thus we have the following system of first order linear ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 6x_1 - 11x_2 + 6x_3 \end{aligned}$$

Translating this into matrix form gives us

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(d) We can verify that each element of S is a solution to $\mathbf{x}' = A\mathbf{x}$ by direct calculation. As an example

$$\begin{bmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \\ 8e^{2t} \end{bmatrix}$$

and

$$A \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \\ 6e^{2t} - 22e^{2t} + 24e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \\ 8e^{2t} \end{bmatrix}$$

And since the Wronskian calculated in part (e) is nonzero, the three elements of S are linearly independent and thus form a fundamental solution set.

(e) Let $\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t)$ be the three functions in S. Then the Wronskian is

$$\begin{aligned} W[\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t)](t) &= \det \begin{bmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{bmatrix} \\ &= e^t(18e^{5t} - 12e^{5t}) - e^{2t}(9e^{4t} - 3e^{4t}) + e^{3t}(4e^{3t} - 2e^{3t}) \\ &= 2e^{6t} \end{aligned}$$

which is just the same as in part (b) (in fact the entire calculation is exactly the same as in part (b)).

Section 9.5

9.5.19 We need to find the eigenvalues and corresponding eigenvectors of A. We first compute the characteristic polynomial.

$$\det(A - \lambda I) = (-1 - \lambda)(1 - \lambda) - 8 = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3)$$

Thus the eigenvalues are 3 and -3. Using row reduction, we can find that corresponding eigenvectors are

$$\begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}$$

So a fundamental matrix is

$$\begin{bmatrix}e^{3t} & e^{-3t}\\4e^{3t} & -2e^{-3t}\end{bmatrix}$$

9.5.21 We need to find the eigenvalues and corresponding eigenvectors of A. We first compute the characteristic polynomial.

$$\det(A-\lambda I) = -\lambda(-\lambda(7-\lambda)+14)-(-8) = -\lambda^3+7\lambda^2-14\lambda+8 = (\lambda-1)(\lambda-2)(4-\lambda)$$

Thus the eigenvalues are $1, 2$ and 4. Using row reduction, we can find that corresponding eigenvectors are

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}1\\4\\16\end{bmatrix}$$

So a fundamental matrix is

$$\begin{bmatrix}e^{t} & e^{2t} & e^{4t}\\e^{t} & 2e^{2t} & 4e^{4t}\\e^{t} & 4e^{2t} & 16e^{4t}\end{bmatrix}$$

9.5.31 Let $A = \left[\begin{smallmatrix}1 & 3\\3 & 1\end{smallmatrix}\right]$. We need to find the eigenvalues and corresponding eigenvectors of A. We first compute the characteristic polynomial.

$$\det(A-\lambda I) = (1-\lambda)(1-\lambda)-9 = \lambda^2-2\lambda-8 = (\lambda-4)(\lambda+2)$$

Thus the eigenvalues are 4 and -2. Using row reduction, we can find that corresponding eigenvectors are

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$$

So a fundamental matrix is

$$\mathbf{X}(t) = \begin{bmatrix}e^{4t} & e^{-2t}\\e^{4t} & -e^{-2t}\end{bmatrix}$$

As shown in exercise 9.4.28, the solution to the initial value problem is

$$\mathbf{X}(t)\mathbf{X}^{-1}(0)\begin{bmatrix}3\\1\end{bmatrix} = \mathbf{X}(t)\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}^{-1}\begin{bmatrix}3\\1\end{bmatrix} = \mathbf{X}(t)\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2e^{4t}+e^{-2t}\\2e^{4t}-e^{-2t}\end{bmatrix}$$

9.5.32 Let $A = \left[\begin{smallmatrix}6 & -3\\2 & 1\end{smallmatrix}\right]$. We need to find the eigenvalues and corresponding eigenvectors of A. We first compute the characteristic polynomial.

$$\det(A-\lambda I) = (6-\lambda)(1-\lambda)+6 = \lambda^2-7\lambda+12 = (\lambda-4)(\lambda-3)$$

Thus the eigenvalues are 4 and 3. Using row reduction, we can find that corresponding eigenvectors are

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So a fundamental matrix is

$$\mathbf{X}(t) = \begin{bmatrix} 3e^{4t} & e^{3t} \\ 2e^{4t} & e^{3t} \end{bmatrix}$$

As shown in exercise 9.4.28, the solution to the initial value problem is

$$\mathbf{X}(t)\mathbf{X}^{-1}(0) \begin{bmatrix} -10 \\ -6 \end{bmatrix} = \mathbf{X}(t) \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -10 \\ -6 \end{bmatrix} = \mathbf{X}(t) \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -12e^{4t} + 2e^{3t} \\ -8e^{4t} + 2e^{3t} \end{bmatrix}$$

9.5.45 First we must formulate the problem as a first order system of ODEs. First observe that if there are $x_1(t)$ kilograms of salt in tank A then every liter of water in tank A has $\frac{x_1(t)}{50}$ kilograms of salt since there are 50 liters total and we have been assured that the water and salt are well mixed. Thus the rate at which salt is leaving tank A at time t is $\frac{4x_1(t)}{50}$ kilograms per minute. By similar reasoning, salt is entering tank A at a rate of $\frac{x_2(t)}{50}$ kilograms per minute, exiting B at a rate of $\frac{4x_2(t)}{50}$ kilograms per minute and entering B at a rate of $\frac{4x_2(t)}{50}$ kilograms per minute. Thus we have the following system of differential equations

$$x_1'(t) = \frac{-4x_1(t)}{50} + \frac{x_2(t)}{50} x_2'(t) = \frac{4x_1(t)}{50} - \frac{4x_2(t)}{50}$$

When written in normal form we have

$$\mathbf{x}'(t) = \begin{bmatrix} -4/50 & 1/50 \\ 4/50 & -4/50 \end{bmatrix} \mathbf{x}(t).$$

As usual, we can solve it by finding the eigenvectors and eigenvalues of the matrix. The eigenvalues are $\frac{-2}{50} = \frac{-1}{25}$ and $\frac{-6}{50} = \frac{-3}{25}$ and corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus a fundamental matrix is

$$\begin{bmatrix} e^{-t/25} & e^{-3t/25} \\ 2e^{-t/25} & -2e^{-3t/25} \end{bmatrix}.$$

Solving for the initial values of $x_1(0) = 2.5$ and $x_2(0) = 0$ we get

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} (2.5/2)(e^{-t/25} + e^{-3t/25}) \\ 2.5(e^{-t/25} - e^{-3t/25}) \end{bmatrix}$$

as our final solution.

And here is a graph of $x_1(t)$ and $x_2(t)$, created with the matplotlib python library: