Problem Set 11 Solutions

November 9, 2016

1 Lay, 7.1

#1 Not symmetric.

#17 We are told the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are 5, 2, and -2. To find an eigenvector for the value 5, we must solve $\begin{bmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix} v_5 = 0$. Using your favorite method, you should find $v_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (or something proportional). Similarly for the eigenvector corresponding to 2, we solve $\begin{bmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix} v_2 = 0$, which gives $v_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$. For -2 we solve $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{bmatrix} v_{-2} = 0$, giving $v_{-2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Putting these together in the usual way, we get $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$. You can check directly that the columns of P (i.e. the eigenvectors of A) are in fact orthogonal.

#26
a. True: this is part (d) of the spectral theorem on page 397.
b. True: to show B is symmetric we must show $B^T = B$. Well, $B^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = B$.
c. False: we just show that if B is orthogonally diagonalizable, it must be symmetric. Well if $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then B is not symmetric, hence not orthogonally diagonalizable; yet it is orthogonal.
d. True: this is part (b) of the spectral theorem.

27 In any case, we will use the fact that a matrix M is symmetric if $M^T = M$. If A is symmetric then $A^T = A$. So firstly, $(B^T AB)^T = B^T A^T (B^T)^T = B^T A (B^T)^T = B^T AB$, hence $B^T AB$ is symmetric. Secondly, $(B^T B)^T = B^T (B^T)^T = B^T B$, so $B^T B$ is symmetric. Lastly $(BB^T)^T = (B^T)^T B^T = BB^T$, so BB^T is symmetric as well.

29 If we write $A = PDP^{-1}$ where $P^{-1} = P^T$, then A being invertible means that D is also invertible. So $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$, and P is still an orthogonal matrix, so writing A^{-1} in this form shows that it is orthogonally diagonalizable.

2 4.1

1 Let $y(t) = \cos\omega t$ with $\omega = \sqrt{k/m}$. For $b = 0$ and $F_{\text{ext}}(t) = 0$, equation (3) reads $my'' + ky = 0$, or $y'' = -\frac{k}{m}y$. Well, the second derivative of $\cos\omega t$, by the chain rule, is $-\omega^2 \cos\omega t = -\frac{k}{m}\cos\omega t = -\frac{k}{m}y$.

2
(a) Suppose $y(t)$ is a solution for $my'' + by' + ky = 0$. Then $m(cy)'' + b(cy)' + k(cy) = cmy'' + cby' + cky = c\big(my'' + by' + ky\big) = c \cdot 0 = 0$.
(b) If $y_1(t), y_2(t)$ are both solutions for $my'' + by' + ky = 0$, then $m(y_1 + y_2)'' + b(y_1 + y_2)' + k(y_1 + y_2) = \Big(my_1'' + by_1' + ky_1\Big) + \Big(my_2'' + by_2' + ky_2\Big) = 0 + 0 = 0$, hence $y_1 + y_2$ is also a solution. Noticed that in parts (a) and (b), we completely used the fact that derivatives are linear.

3 4.2

3 First we solve the associated auxiliary equation, $r^2 - r - 2 = 0$. This factors as $(r - 2)(r + 1) = 0$, so $r = 2$ or $r = -1$. Then we know that the general solution is $y(t) = C_1e^{2t} + C_2e^{-t}$.

5 The associated auxiliary equation is $r^2 - 5r + 6 = 0$. This factors as $(r - 2)(r - 3) = 0$, so $r = 2$ or $r = 3$. Then the general solution is $y(t) = C_1e^{2t} + C_2e^{3t}$.

10 The associated auxiliary equation is $r^2 - r - 11 = 0$, whose solutions are $r = \frac{1 \pm \sqrt{45}}{2} = \frac{1 \pm 3\sqrt{5}}{2}$. So the general solution is $y(t) = C_1e^{(1+3\sqrt{5})/2 \cdot t} + C_2e^{(1-3\sqrt{5})/2 \cdot t}$.

13 First let us find the general solution: the associated auxiliary equation is $r^2 + 2r - 8 = (r+4)(r-2) = 0$, so $r = 2$ or $r = -4$. So the *general* solution is $y(t) = C_1e^{2t} + C_2e^{-4t}$. Then the initial conditions tell us: $y(0) = 3 = C_1 + C_2$ and $y'(0) = -12 = 2C_1 - 4C_2$. Solving this system of equations however you like, we find $C_1 = 0, C_2 = 3$, and so $y(t) = 3e^{-4t}$. This is the solution to the initial value problem.

4 4.3

3 The auxiliary equation is $r^2 - 10r + 26 = 0$. You can use the quadratic equation: for this equation, I think it is easier to complete the square. So $(r - 5)^2 + 1 = 0$, so $r = 5 \pm i$. Using the complex conjugate roots formula on page 169, we see then that the general solution is $y(t) = e^{5t}\Big(C_1 \sin t + C_2 \cos t\Big)$.

13 The auxiliary equation is $r^2 + 2r + 5 = 0$. Completing the square here gives $(r+1)^2 + 4 = 0$, so $r = -1 \pm 2i$ (again, you could also use quadratic equation to find this). So by the same formula on p. 169, the general solution is $y(t) = e^{-t}\Big(C_1 \sin 2t + C_2 \cos 2t\Big)$.

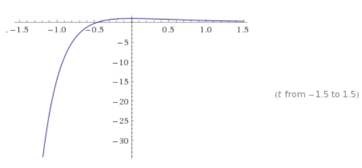
22 First we find the general solution: the auxiliary equation is $r^2 + 2r + 17 = 0$, and this has roots $r = -1 \pm 4i$. So the general solution is $y(t) = e^{-t}\Big(C_1 \sin 4t + C_2 \cos 4t\Big)$. Then $y(0) = C_2 = 1$, and $y'(0) = -C_2 + 4C_1 = -1$, hence $C_1 = 0$. So the solution to the initial value problem is $y(t) = e^{-t} \cos 4t$.

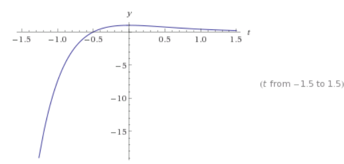
28 For $b = 5$, the auxiliary equation is $r^2 + 5r + 4 = (r + 4)(r + 1) = 0$, so $r = -4, -1$ and the general solution is $y(t) = C_1 e^{-4t} + C_2 e^{-t}$. Then $y(0) = C_1 + C_2 = 1$ and $y'(0) = -4C_1 - C_2 = 0$, so $C_1 = -\frac{1}{3}, C_2 = \frac{4}{3}$, and $y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$.

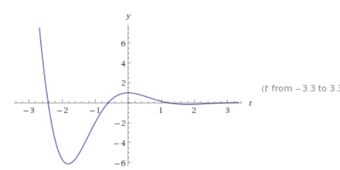
For $b = 4$, the auxiliary equation is $r^2 + 4r + 4 = (r + 2)^2 = 0$, so the general solution is $y(t) = C_1 e^{-2t} + C_2 t e^{-2t}$. Then $y(0) = C_1 = 1$ and $y'(0) = -2C_1 + C_2 = 0$ hence $C_2 = 2$ so $y(t) = (1 + 2t)e^{-2t}$.

Finally for $b = 2$ the auxiliary equation is $r^2 + 2r + 4 = 0$ and the roots are $r = -1 \pm i \cdot \sqrt{3}$, so the general solution is $y(t) = e^{-t}\big(C_1 \sin \sqrt{3}t + C_2 \cos \sqrt{3}t\big)$. Then $y(0) = C_2 = 1$ and $y'(0) = -C_2 + \sqrt{3}C_1 = 0$, so $C_1 = \frac{1}{\sqrt{3}}$, and $y(t) = e^{-t}\big(\frac{1}{\sqrt{3}} \sin \sqrt{3}t + \cos \sqrt{3}t\big)$.

Here are graphs of the three solutions, via Wolfram Alpha:







30 For $y(t) = e^{rt}$, we have that $y'(t) = re^{rt}$, even when r is complex. For $r = \alpha + i\beta$, this gives us formula (7) on page 168, as desired.

32
(a) So first we wish to solve $10y'' + 250y = 0$, i.e. $y'' + 25y = 0$. The auxiliary equation is $r^2 + 25 = 0$, so $r = \pm 5i$, so the general solution is $y(t) = C_1 \sin 5t + C_2 \cos 5t$. Then $y(0) = C_2 = 0.3, y'(0) = 5C_1 = -0.1$, so $C_1 = -0.02$, and $y(t) = 0.3 \cos 5t - 0.02 \sin 5t$.
(b) There $\beta = 5$, so the frequency is $\frac{5}{2\pi}$.

36
(a) The roots of the auxiliary equation for problem 21 are still $-1 \pm i$, so now we'll write the general solution as $y(t) = d_1 e^{(-1+i)t} + d_2 e^{(-1-i)t}$. Now the initial values give: $y(0) = 2 = d_1 + d_2, y'(0) = 1 = (-1 + i)d_1 + (-1 - i)d_2$. Solving this system of linear equations with *complex* coefficients any way you like, we find that $d_1 = 1 - \frac{3}{2}i$ and $d_2 = 1 + \frac{3}{2}i$.
(b) (Notice d_1 and d_2 in part (a) were complex conjugates.) We can actually just plug in $t = 0$ to prove part (b): if d_1 and d_2 are not complex conjugates, then $y(0) = d_1 + d_2$ is not real, hence the solution y is not real (a solution being real means that its values are real for all t).

5 4.4

1 No: the method of undetermined coefficients requires that the power of t in the inhomogeneous term be *nonnegative*, but the power here is -1.

2 Yes: the inhomogeneous term $t^3 \cos 4t$ is of the form in equation 15 on page 180.

3 Yes: $(\sin x)/e^{4x} = \sin x \cdot e^{-4x}$ so this is also of the form of equation 15.

11 We saw in the book that when the inhomogeneous term is an exponential, e^{rt}, we guess $y(t) = Ae^{rt}$ as a particular solution and then solve for A (assuming r is not a root of the auxiliary equation, in which case we need to use a different guess). Well, $2^x = (e^{\ln 2})^t = e^{t \ln 2}$, so we should guess $y(t) = A \cdot 2^x = Ae^{t \ln 2}$. Plugging this in to the left hand side, $y''(x) + y(x) = A(\ln 2)^2 2^x + A2^x = 2^x$, hence $A(\ln 2)^2 + A = 1$, i.e. $A = \frac{1}{(\ln 2)^2+1}$.

12 By undetermined coefficients, we guess $x(t) = At^2 + Bt + C$, i.e. our guess is a quadratic polynomial. Plugging this into the left side, we get $2x' + x = At^2 + (B + 2A)t + (C + B)$, and the right side is still $3t^2$. Thus comparing the t^2-coefficients we see $A = 3$; comparing the t-coefficients we see $B + 2A = 0$, i.e. $B = -6$; comparing constant coefficients we see $C + B = 0$, i.e. $C = 6$. Thus our particular solution is $x(t) = 3t^2 - 6t + 6$.

21 We must use the special case in the table on page 180 this time: the auxiliary equation of the left side is $r^2 - 4r + 4 = (r - 2)^2 = 0$, and the inhomogeneous term is te^{2t}. Thus, in the notation of the table given, $s = 2$, so our guess must be $x(t) = t^2(At + B)e^{2t} = (At^3 + Bt^2)e^{2t}$.

Working it out gives $x'(t) = \Big(2At^3 + (2B + 3A)t^2 + 2Bt\Big)e^{2t}$ and $x''(t) = \Big(4At^3 + (4B + 12A)t^2 + (8B + 6A)t + 2B\Big)e^{2t}$. Now let us plug $x(t)$ into the left side of our equation: we should hope that if our guess works, the t^3 and t^2 coefficients will cancel, so that we have 2 equations for our 2 unknowns. If you work out the arithmetic, you see this in fact happens: $x''(t) - 4x'(t) + 4x(t) = (6At + 2B)e^{2t}$. Comparing with te^{2t}, we find that $A = \frac{1}{6}, B = 0$, and so our particular solution is $\frac{1}{6}t^3 e^{2t}$.

29 We see, just like above, that 3 is a double root of the auxiliary equation of the left hand side: the equation is $r^2 - 6r + 9 = (r - 3)^2 = 0$. So in the notation of the table on page 180 again, $s = 2$ and $m = 6, r = 3$. So our guess would have to be $y(t) = t^2\Big(At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft + G)e^{3t}$.

6 4.5

2
(a) Superposition tells us that $5y_1(t) = 5 \cos t$ is a particular solution.
(b) Superposition tells us that $y_1(t) - 3y_2(t) = \cos t - e^{2t}$ is a particular solution.
(c) This time, $4y_1(t) + 18y_2(t) = 4 \cos t + 6e^{2t}$ is a particular solution.

5 Let us find the homogeneous solution, i.e. the general solution to $y'' + 5y' + 6 = 0$. Solving just as in section 4.2, we get $C_1 e^{-2t} + C_2 e^{-3t}$. Since we know, by the superposition principle, that the general solution is the sum of the general homogeneous solution and a particular solution, we see that the general solution here is $y(t) = C_1 e^{-2t} + C_2 e^{-3t} + e^x + x^2$.

7 Write this equation as $y'' - 2y = 2\tan^3 x$, so that we can see the homogeneous equation is $y'' - 2y = 0$. Again, as in 4.2, the general solution to the latter is $y = C_1 \sin\sqrt{2}t + C_2 \cos\sqrt{2}t$, so the general solution to the original inhomogeneous equation is $y(t) = C_1 \sin\sqrt{2}t + C_2 \cos\sqrt{2}t + \tan x$.

18 First let us find a particular solution: undetermined coefficients tells us to guess $y(t) = At^2 + Bt + C$; then $y'(t) = 2At + B$ and $y''(t) = 2A$. Plugging this into the left gives $y'' - 2y' - 3y = 2A - (2At + B) - 3(At^2 + Bt + C) = 3t^2 - 5$. Comparing t^2 coefficients, we see $A = -1$. Then comparing t coefficients, $2 - 3B = 0$ hence $B = \frac{2}{3}$; then finally $C = \frac{7}{9}$. So our particular solution is $y_p(t) = -t^2 + \frac{2}{3}t + \frac{7}{9}$.

Then the auxiliary equation for the homogeneous part is $r^2 - 2r - 3 = (r-3)(r+1) = 0$, so the general homogeneous solution is $C_1e^{3t} + C_2e^{-t}$. Thus the general solution is $y(t) = C_1e^{3t} + C_2e^{-t} - t^2 + \frac{2}{3}t + \frac{7}{9}$.

20 Method of undetermined coefficients tells us our guess should be $y(t) = A\sin\theta + B\cos\theta$. Plugging this in to the left side gives $y''(\theta) + 4y(\theta) = (-A\sin\theta - B\cos\theta) + 4(A\sin\theta + B\cos\theta) = 3(A\sin\theta + B\cos\theta)$, and comparing this to the inhomogeneous part, $\sin\theta - \cos\theta$, we find $A = \frac{1}{3}, B = -\frac{1}{3}$.

For the homogeneous part, by the same methods as 4.2 we see the general solution is $C_1 \sin 2\theta + C_2 \cos 2\theta$. Thus the general solution to $y''(\theta) + 4y(\theta) = \sin\theta - \cos\theta$ is $y(t) = C_1 \sin 2\theta + C_2 \cos 2\theta + \frac{1}{3}(\sin\theta - \cos\theta)$.