MATH 54 - Homework 10 Solutions

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6.7.1
||x|| = √9 = 3; ||y|| = √105; |⟨x, y⟩|^2 = (20 − 5)^2 = 225.

6.7.7
proj_{Span(p)}q = \langle q, p \rangle \langle p, p \rangle^{-1} p, so we need to calculate \langle p, q \rangle and \langle p, p \rangle.
\langle p, q \rangle = p(−1)q(−1) + p(0)q(0) + p(1)q(1) = 3 + 20 + 5 = 28;
\langle p, p \rangle = 9 + 16 + 25 = 50.
So proj_{Span(p)}q = \frac{60}{25} + \frac{14}{25}t.

6.7.9 (b) (not required, but used in the next exercise)
We want to apply Gram-Schmidt to \{1, t, t^2\}. Notice \{1, t\} = 0, so we only need to do calculate:
q' = t^2 - \frac{(t^2, 1) \langle 1, 1 \rangle (1, 1)}{(1, 1)^2 - \langle 1, 1 \rangle^2}.
\langle 1, 1 \rangle = (−3)^2 + (−1)^2 + 12 + 3^2 = 20;
\langle t, 1 \rangle = 4; \langle t^2, t \rangle = −27 − 1 + 1 + 27 = 0.
So we get q' = t^2 − 5.
The last part of the question asks us to scale q' to q so that q(−3) = q(3) = 1 and q(−1) = q(1) = −1. It’s
easy to see that dividing q' by 4 does this.
So q = \frac{1}{4}(t^2 − 5).

6.7.10
By the previous exercise (with q as above), \{1, t, q\} is orthogonal, so the approximation is
proj_{Span(1, t, q)}t^3 = \frac{\langle t^3, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle t^3, t \rangle}{\langle t, t \rangle} t + \frac{\langle t^3, q \rangle}{\langle q, q \rangle} q.
\langle t^3, 1 \rangle = (−3)^3 + (−1)^3 + 13 + 3^3 = 0;
\langle t^3, t \rangle = (−3)^4 + (−1)^4 + 1^4 + 3^4 = 164;
\langle t, t \rangle = (−3)^2 + (−1)^2 + 1^2 + 3^2 = 20;
\langle t^3, q \rangle = (−3)^3 q(−3) + (−1)^3 q(−1) + 1^3 q(1) + 3^3 q(3) = −27 + 1 − 1 + 27 = 0.
So proj_{Span(1, t, q)}t^3 = \frac{14}{9} q.

6.7.14
Let T : V \to \mathbb{R}^n a one-to-one linear transformation. We check that \langle u, v \rangle := T(u) \cdot T(v) is an inner product.
In checking each property, we use the corresponding fact about the dot product of \mathbb{R}^n.

1. \langle u, v \rangle = T(u) \cdot T(v) = T(v) \cdot T(u) = \langle v, u \rangle

2. \langle u + v, w \rangle = T(u + v) \cdot T(w) = T(u) + T(v) = T(w) = T(v) \cdot T(u) = \langle w, u \rangle + \langle v, u \rangle (we used
the linearity of T in the second step)

3. \langle cu, v \rangle = T(cu) \cdot T(v) = cT(u) \cdot T(v) = c(T(u) \cdot T(v)) = c\langle u, v \rangle (again, we used linearity of T in the
second step)

4. \langle u, u \rangle = T(u) \cdot T(u) \geq 0 and \langle u, u \rangle = 0 if T(u) \cdot T(u) = 0 if T(u) = 0 if u = 0, (using in this last step
that T is linear and one-to-one ).
6.7.16

Suppose \( \{u, v\} \) is an orthonormal set.
\[
\|u - v\|^2 = \langle u - v, u - v \rangle = \langle u, u - v \rangle - \langle v, u - v \rangle = \langle u - v, u \rangle - \langle v, u \rangle - \langle v, u \rangle + \langle v, v \rangle = \|u\|^2 - 2 \langle u, v \rangle + \|v\|^2.
\]
Since \( \{u, v\} \) is an orthonormal set, \( \|u\| = \|v\| = 1 \) and \( \langle u, v \rangle = 0 \). So \( \|u - v\|^2 = 1 - 0 + 1 = 2 \).
Hence \( \|u - v\| = \sqrt{2} \).

6.7.18

From 6.7.16, \( \|u - v\|^2 = \|u\|^2 - 2 \langle u, v \rangle + \|v\|^2 \). A similar calculation shows \( \|u + v\|^2 = \|u\|^2 + 2 \langle u, v \rangle + \|v\|^2 \).
So \( \|u + v\|^2 + \|u - v\|^2 = 2 \|u\|^2 + 2 \|v\|^2 \).

6.7.22

\[ \langle f, g \rangle = \int_0^1 (5t - 3)(t^3 - t^2)dt = \int_0^1 5t^4 - 8t^3 + 3t^2\, dt = \left( t^5 - 2t^4 + t^3 \right) \bigg|_0^1 = 0 \]

6.7.26

\( \{1, t\} = \int_{-2}^{2} t \, dt = (\frac{1}{2} t^2) \bigg|_{-2}^{2} = 0 \), so 1 and \( t \) are already orthogonal.
We need only apply Gram-Schmidt to \( t^2 \).
\[
\langle 1, t^2 \rangle = (\frac{1}{2} t^3) \bigg|_{-2}^{2} = \frac{16}{3},
\]
\[
\langle 1, 1 \rangle = t^2 \bigg|_{-2}^{2} = 4,
\]
\[
\langle t, t^2 \rangle = \frac{1}{2} t^4 \bigg|_{-2}^{2} = 0.
\]
So \( \{1, t, t^2 - \frac{4}{3}\} \) is our desired orthogonal basis.