MATH 54 - Homework 10 Solutions

10/26

6.7.1

$\|x\| = \sqrt{9} = 3$; $\|y\| = \sqrt{105}$; $|\langle x, y\rangle|^2 = (20-5)^2 = 225$.

6.7.7

$proj_{\text{Span}\{p\}} q = \frac{\langle q,p\rangle}{\langle p,p\rangle} p$, so we need to calculate $\langle p, q\rangle$ and $\langle p, p\rangle$.
$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) = 3 + 20 + 5 = 28$;
$\langle p, p\rangle = 9 + 16 + 25 = 50$.
So $proj_{\text{Span}\{p\}} q = \frac{56}{25} + \frac{14}{25} t$.

6.7.9 (b) (not required, but used in the next exercise)

We want to apply Gram-Schmidt to $\{1, t, t^2\}$. Notice $\langle 1, t\rangle = 0$, so we only need to do calculate:
$q' = t^2 - \frac{\langle t^2,1\rangle}{\langle 1,1\rangle} 1 - \frac{\langle t^2,t\rangle}{\langle t,t\rangle} t$.
$\langle t^2, 1\rangle = (-3)^2 + (-1)^2 + 1^2 + 3^2 = 20$;
$\langle 1, 1\rangle = 4$; $\langle t^2, t\rangle = -27 - 1 + 1 + 27 = 0$.
So we get $q' = t^2 - 5$.
The last part of the question asks us to scale q' to q so that $q(-3) = q(3) = 1$ and $q(-1) = q(1) = -1$. It's easy to see that dividing q' by 4 does this.
So $q = \frac{1}{4}(t^2 - 5)$.

6.7.10

By the previous exercise (with q as above), $\{1, t, q\}$ is orthogonal, so the approximation is

$$proj_{\text{Span}\{1,t,q\}} t^3 = \frac{\langle t^3, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle t^3, t\rangle}{\langle t, t\rangle} t + \frac{\langle t^3, q\rangle}{\langle q, q\rangle} q.$$

$\langle t^3, 1\rangle = (-3)^3 + (-1)^3 + 1^3 + 3^3 = 0$;
$\langle t^3, t\rangle = (-3)^4 + (-1)^4 + 1^4 + 3^4 = 164$;
$\langle t, t\rangle = (-3)^2 + (-1)^2 + 1^2 + 3^2 = 20$;
$\langle t^3, q\rangle = (-3)^3 q(-3) + (-1)^3 q(-1) + 1^3 q(1) + 3^3 q(3) = -27 + 1 - 1 + 27 = 0$.
So $proj_{\text{Span}\{1,t,q\}} t^3 = \frac{41}{5} t$.

6.7.14

Let $T : V \to \mathbb{R}^n$ a one-to-one linear transformation. We check that $\langle u, v\rangle := T(u) \cdot T(v)$ is an inner product. In checking each property, we use the corresponding fact about the dot product of $\mathbb{R}^n$.

1. $\langle u, v\rangle = T(u) \cdot T(v) = T(v) \cdot T(u) = \langle v, u\rangle$
2. $\langle u + v, w\rangle = T(u + v) \cdot T(w) = \big(T(u) + T(v)\big) \cdot T(w) = T(u) \cdot T(w) + T(v) \cdot T(w) = \langle u, w\rangle + \langle v, w\rangle$ (we used the linearity of T in the second step)
3. $\langle cu, v\rangle = T(cu) \cdot T(v) = cT(u) \cdot T(v) = c\big(T(u) \cdot T(v)\big) = c\langle u, v\rangle$ (again, we used linearity of T in the second step)

4 $\langle u, u\rangle = T(u) \cdot T(u) \geq 0$ and $\langle u, u\rangle = 0$ iff $T(u) \cdot T(u) = 0$ iff $T(u) = 0$ iff $u = 0$, (using in this last step that T is linear and one-to-one).

6.7.16

Suppose $\{u, v\}$ is an orthonormal set.

$||u - v||^2 = \langle u - v, u - v\rangle = \langle u, u - v\rangle - \langle v, u - v\rangle = \langle u - v, u\rangle - \langle u - v, v\rangle = \langle u, u\rangle - \langle v, u\rangle - \langle v, u\rangle + \langle v, v\rangle = ||u||^2 - 2\langle u, v\rangle + ||v||^2$.

Since $\{u, v\}$ is an orthonormal set, $||u|| = ||v|| = 1$ and $\langle u, v\rangle = 0$. So $||u - v||^2 = 1 - 0 + 1 = 2$.

Hence $||u - v|| = \sqrt{2}$.

6.7.18

From 6.7.16, $||u - v||^2 = ||u||^2 - 2\langle u, v\rangle + ||v||^2$. A similar calculation shows $||u + v||^2 = ||u||^2 + 2\langle u, v\rangle + ||v||^2$.

So $||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$.

6.7.22

$\langle f, g\rangle = \int_0^1 (5t - 3)(t^3 - t^2)dt = \int_0^1 5t^4 - 8t^3 + 3t^2 dt = (t^5 - 2t^4 + t^3)|_0^1 = 0$

6.7.26

$\langle 1, t\rangle = \int_{-2}^2 t dt = (\frac{1}{2}t^2)|_{-2}^2 = 0$, so 1 and t are already orthogonal.

We need only apply Gram-Schimdt to t^2.

$\langle 1, t^2\rangle = (\frac{1}{3}t^3)|_{-2}^2 = \frac{16}{3}$,

$\langle 1, 1\rangle = t|_{-2}^2 = 4$,

$\langle t, t^3\rangle = \frac{1}{5}t^4|_{-2}^2 = 0$,

So $\{1, t, t^2 - \frac{4}{3}\}$ is our desired orthogonal basis.