The Final exam will be cumulative and will cover everything in the course so far (Lay Chaps 1-7 assigned sections, Nagle Saff and Snider Chaps 4, 9, 10). You should know all the definitions and theorems we have covered in lecture. Knowing a definition means you should be able to state it precisely, and you should know how to go about checking whether a given object satisfies the definition or not. Knowing a theorem means you should know the statement of the theorem and how to use it to solve problems (but not necessarily the proof — except for starred theorems from this and previous review sheets).

Here is the new content (post Midterm 2) which will be on the final. The main resource for this section is my lecture notes on bCourses, which present the material slightly differently from the book, and the order below follows the order in my notes. But you can use the book as a resource as well. Perhaps the most useful parts of the book are the chapter end summaries (chapters 4, 9, 10).

Key definitions, theorems, and concepts:

- Abstract inner product spaces and their basic properties (Lay 6.7).
- Spectral Theorem: if $A = A^T$ then A is diagonalizable with orthogonal eigenvectors and real eigenvalues (Lay Ch 7).
- First and Second Order Linear ODE (Ordinary Differential Equation).
- Homogeneous and Nonhomogeneous equation, relationship between their solution spaces.
- Vector spaces of functions associated with the above. How to view them as $T(y) = f$ for a linear operator[1]. $T = ad^2/dx^2 + bd/dx + cI$.
- Initial value problem for ODE with initial conditions. Existence and uniqueness of the solution.
- Auxilliary equation, general solution depending on roots of the auxilliary equation: $e^{r_1 t}, e^{r_2 t}$ or e^{rt}, te^{rt}.
- Complex exponentials, relation to sin and cos via Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
- How to extract real solutions from a complex solution using the above.
- Wronskian lemma for second order ODE.
- Method of Undetermined Coefficients *only for polynomials and exponentials*, i.e. $f(t) = t^m$ and $f(t) = e^r t, e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)$ (the last two follow from the case $e^{\alpha+i\beta}t$). In particular, the form $f(t) = t^m e^{rt}$ will not be on the exam.
- Superposition principle.
- "3-step method" for second order ODE.
- Normal form of a system of first order ODE.
- Reduction from higher order ODE to systems of first order ODE.
- Vector spaces of functions $f : \mathbb{R} \to \mathbb{R}^n$ or "vectors of functions" associated with the above. How to view it as $T(y) = f$ for an appropriate linear transformation T.
- Solution to $y'(t) = Ay(t)$ when A is diagonalizable, via eigenvectors of A or via change of variables through diagonalization.
- How to find real solutions from solutions to the above arising from *complex* eigenvalues of A.
- Fundamental Solution Set of a system in normal form.
- "3-step method" for a system in normal form with initial conditions. Existence and uniqueness theorem for the initial value problem.
- Method of undetermined coeffs for systems of ODE in the special cases $f(t) = e^{rt}v$ for some vector v or $f(t) =$ a vector of polynomials.
- Wronskian Lemma for systems of ODE.
- Boundary value problem for ODE.
- Partial derivative, linearity of partial derivative.
- Heat Equation (only the case with $u(0,t) = u(L,t) = 0$).
- Eigenvectors (i.e., "eigenfunctions") and eigenvalues of the second derivative operator $\beta\partial/\partial x^2$.
- Solution of the heat equation for $f(x) = C\sin(n\pi x/L)$.
- Solution of the heat equation for general $f(x)$ using Fourier Sine Series.
- Inner Product space associated with Fourier Sine Series, orthogonality properties of sines.
- Integral formula for Fourier coefficients.
- Inner Product interpretation of Fourier Sine Coefficients.

[1] Note that linear transformations from a vector space to itself are called operators.

- (Full) Fourier Series (i.e., Cosine and Sine) on the interval $[-\pi, \pi]$.
- Convergence of Fourier series (assuming f' piecewise continuous).

Types of Problems. The problems will be similar to the homework problems, but to help you study, here are the main kinds:

- Decide whether elements of a given abstract inner product space are orthogonal, and compute orthogonal projections in such spaces.
- Find the general solution to a given second order ODE or system of first order ODE (both homogeneous and inhomogeneous cases).
- Find a solution to the above which fits given initial conditions / boundary conditions.
- Reduce a given higher order ODE to a system of first order ODE.
- Reason about linear independence of functions using the Wronskian lemma. Decide whether a given set of functions can be a set of solutions to an ODE or system of ODE.
- Compute the Fourier coefficients (both for Sine series and Sine and Cosine series) of a given function.
- Find a solution to the heat equation given initial data $f(x)$.
- Find the eigenvalues and eigenvectors (i.e., "eigenfunctions") of a given differential operator, subject to boundary conditions.

This list is not comprehensive. I will most likely give you problems that involve combining ideas from multiple chapters. I will NOT ask you to give proofs of any of the theorems we covered after the second midterm. I may ask you to prove any starred theorems from the review sheets for Midterm 1 and Midterm 2.

Trigonometry. Many people asked which trig identities are required. The only ones you need to know are the sum to product formulas:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

and

$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b).$$

A good exercise is: derive these using Euler's identity (hint: use the fact that $e^{i(a+b)} = e^{ia}e^{ib}$.

By applying the above to $\cos(a-b)$ and $\sin(a-b)$, we obtain also

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

and

$$\sin(a-b) = -\cos(a)\sin(b) + \sin(a)\cos(b).$$

By adding and subtracting the above, you can obtain product to sum formulas such as

$$\sin(a)\sin(b) = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right),$$

which are useful in showing that the sine functions are orthogonal with respect to the integration inner product. Thus, it is also good to remember these (or at least know how to derive them in case you need them).

Separation of Variables. I did not discuss separation of variables in class, and instead I presented the solution of the heat equation in terms of eigenfunctions of $\partial^2/\partial x^2$. However, the textbook and well as several GSIs used separation of variables, which involves making the "guess" $u(x,t) = X(x)T(t)$ and solving for X and T. While I do prefer the method I presented in class, you are free to use this method to solve problems on the exam.

NOT ON THE EXAM. Wave equation, matrix exponential, separation of variables, undetermined coeffs for $t^m e^{rt}$, delicate convergence properties of Fourier series (beyond what we did in class).

SHOW YOUR WORK. If you just write a numerical answer without a clear description of the process you used to get it, you will only get partial credit. If something really is obvious (such as saying that the vector $(1,1)$ is orthogonal to $(1,-1)$) use the words "observe that" or "by inspection" to indicate this.