Math 54 Fall 2016 Practice Final

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180 minutes, closed book, closed notes

1. (20 pts) True or False (no need for justification):

   (a) If $AB = 0$ for two square matrices A and B then either $A = 0$ or $B = 0$.

   (b) If A is a square invertible matrix then A and A^{-1} have the same rank.

   (c) If A and B are square and invertible then AB and BA have the same eigenvalues.

   (d) If every entry of a square matrix A is nonzero, then $\det(A) \neq 0$.

   (e) The sum of two diagonalizable matrices must be diagonalizable.

   (f) If A is an $m \times n$ matrix then the rank of $A^T A$ is equal to the rank of A.

   (g) If $A = A^T$ and the only eigenvalue of A is $\lambda = 1$, then $A = I$.

   (h) Any two orthogonal vectors in an inner product space must be linearly independent.

   (i) Suppose W is a subspace of $\mathbb{R}^n$. If $v_1, \ldots, v_k$ is a basis for W and $u_1, \ldots, u_\ell$ is a basis for $W^\perp$ then $v_1, \ldots, v_k, u_1, \ldots, u_\ell$ must be a basis for $\mathbb{R}^n$.

   (j) Two real-valued functions $y_1(t)$ and $y_2(t)$ are linearly independent if and only if their Wronskian determinant is nonzero everywhere.

2. (20 pts) For each of the following, either find an example (and explain why it has the property) or explain why no such example exists.

   - A differential operator $T = a(d^2/dx^2) + b(d/dx) + cI$ with $a \neq 0$ on the vector space
     $$V = \{f : \mathbb{R} \to \mathbb{R},\ f \text{ is infinitely differentiable}\}$$
     which is one to one.
   - A second order linear differential equation with constant coefficients which has $y(t) = e^t$ and $y(t) = \sin(t)$ among its solutions.
   - A 2×2 real matrix A such that the system of ODE $y'(t) = Ay(t)$, $y : \mathbb{R} \to \mathbb{R}^2$, has a fundamental matrix
     $$\begin{bmatrix} -e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}.$$

- A square matrix A such that A is *not diagonal* and $A^2 = A$.
- Two linearly *independent* vector-valued functions $y_1 : \mathbb{R} \to \mathbb{R}^2$ and $y_2 : \mathbb{R} \to \mathbb{R}^2$ such that the vectors $y_1(0)$ and $y_2(0)$ are linearly *dependent* in $\mathbb{R}^2$.

3. (6 pts) Let

$$V = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix} \right\}, W = span\left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

be subspaces of $\mathbb{R}^4$. Find a nonzero vector in $V \cap W$ (i.e., which is in both subspaces).

4. (8 pts) Consider the vector space of 2×2 real matrices with entrywise addition:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

and consider the function $T : V \to V$ defined by

$$T(X) = X + X^T.$$

(a) Show that T is a linear transformation.
(b) Find a basis for $\mathrm{Ker}(T)$.
(c) Find a basis for $\mathrm{Im}(T)$.
(d) Find an eigenvector of T, along with the corresponding eigenvalue.

5. (6 pts) For which real values of a is the matrix

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

diagonalizable? For which a is it invertible?

6. (7 pts) Let V be the vector space of all real valued continuous functions on the interval $[0, 1]$, and consider the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Find a nonzero function in V which is orthogonal to the functions x and x^2, with respect to this inner product.

7. (10 pts) (a) Find a basis of real solutions to the homogeneous differential equation

$$y''(t) - 2y'(t) + 2y(t) = 0.$$

(b) Find the general solution to the inhommogeneous equation

$$y''(t) - 2y'(t) + 2y(t) = t^2 + e^t.$$

(c) Find a solution to (b) satisfying the initial conditions $y(0) = 1$ and $y'(0) = 2$.
(d) Write the equation in (a) as $T_1 \circ T_2(y) = 0$ for two *first order* differential operators T_1 and T_2.

8. (8 pts) Find functions $y_1(t)$ and $y_2(t)$ such that

$$y_1' = -2y_1 + 2y_2 \qquad y_2' = 2y_1 + y_2$$

and $y_1(0) = -1, y_2(0) = 3$.

9. (7 pts) Find a real-valued function $u : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ which satisfies the heat equation

$$\frac{\partial}{\partial t} u(x,t) = 2\frac{\partial^2}{\partial x^2} u(x,t) \qquad u(0,t) = u(\pi,t) = 0,$$

for all $t > 0$, as well as the initial condition

$$u(x,0) = \sin(3x) - \sin(5x).$$

10. (8 pts) Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ defined by $f(x) = |\sin(x)|$. Draw a sketch of the function. Find coefficients a_n, b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_0 \cos(nx) + b_n \sin(nx).$$

(hint: use the product to sum trig formulas)