Math 53 Practice Final, Spring 2018

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180 minutes

1. True or False.

   (a) There are two unit vectors u and v such that the sum $u + v$ has length $1/3$.

   (b) If $f(x, y)$ is continuous and both f_x and f_y are defined and continuous on $\mathbb{R}^2$, then $f(x, y)$ must be differentiable on $\mathbb{R}^2$.

   (c) The work done by a vector field on a particle moving along a parameterized curve C is independent of the time taken to traverse C, and depends only on the trajectory.

   (d) The number of critical points of a differentiable function on $\mathbb{R}^2$ must be finite.

   (e) If $f(x, y, z)$ is a solution of Laplace's equation

   $$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0$$

   then the flux of ∇f through the unit sphere, outwardly oriented, must be zero.

   (f) If $\mathbf{F}$ is a conservative vector field then $\text{div}(F) = 0$.

   (g) There exists a vector field $\mathbf{F}$ such that $\text{div}(F) = x^2 + y^2 + z^2$.

   (h) There exists a vector field $\mathbf{F}$ such that $\text{curl}(F) = \langle x^2, y^2, z^2\rangle$.

   (i) If $\mathbf{F} = \langle 1/3, 1/3, 1/3\rangle$ then the flux of $\mathbf{F}$ across any oriented surface cannot be larger than its surface area.

   (j) If the flux of $\mathbf{F} = \langle P, Q\rangle$ across every closed curve in the plane is zero, then $\mathbf{F}$ must be conservative.

2. A particle moves along the intersection of the surfaces

   $$x^2 + y^2 + 2z^2 = 4, \qquad z = xy.$$

   Let $\langle x(t), y(t), z(t)\rangle$ denote the location of the particle at time t. Suppose that $\langle x(0), y(0), z(0)\rangle = \langle 1, 1, 1\rangle$ and $x'(0) = 1$. Calculate $y'(0)$ and $z'(0)$.

3. Suppose f is a function on $\mathbb{R}^2$ satisfying the following conditions on its directional derivatives:

   $$D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(x, y) = \sqrt{2}x, \qquad D_{\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle} f(x, y) = \sqrt{2}y.$$

   (a) Find $f_x(x, y)$ and $f_y(x, y)$. (b) Assuming that $f(0, 0) = 0$, find the function $f(x, y)$.

4. Suppose that x, y, z are constrained by the equation $g(x, y, z) = 3$. Assume that at the point $P(0, 0, 0)$ we have $g = 3$ and $\nabla g = \langle 2, -1, -1\rangle$. The equation $g = 3$ implicitly defines z as a function of x and y in a neighborhood of the origin. Find the value of $\partial z/\partial x$ at P.

5. (a) Find the equation of a tangent plane to the surface S given by $4xy - z^2 = 0$ at $P(1, 1, 2)$. (b) Use this to approximate the value of $4 \times 1.001 \times .99 - 2.001^2$. (c) Find a parametric equation for the line through P perpendicular to S at P.

6. Use Lagrange multipliers to find the point on the surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$ where the function $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximized.

7. Classify the critical points of the *area 51* function

$$f(x, y) = x^{51} - 51x - y^{51} + 51y$$

using the second derivative test. The reason why this function was chosen is classified.

8. Evaluate by changing the order of integration:

$$\int_0^3 \int_{x^2}^9 xe^{-y^2} dydx.$$

9. Consider the ellipse E given by:

$$x^2/a^2 + y^2/b^2 = 1.$$

(a) Define a change of variables mapping the unit disk $\{u^2 + v^2 \leq 1\}$ to E. (b) Use this this to show that the area of E is πab.

10. Find the volume of the region consisting of all points that are inside the sphere $x^2 + y^2 + z^2 = 4$, above the plane $z = 0$, and below the plane $z = x$.

11. The force exerted by an electric charge at the origin on an electron at the point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = -K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. Find the work done by this force as the electron moves along a straight line segment from $(2, 0, 0)$ to $(2, 1, 5)$.

12. Consider a surface S in 3−space given by an equation $z = f(x)$ (so that its trace in every plane $y = c$ is exactly the same). Show that if $\mathbf{F} = \langle x^2, y^2, xz \rangle$ then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed curve C lying on the surface S. (hint: Stokes' theorem)

13. Let $\mathbf{F} = \langle ay^2, 2y(x+z), by^2 + z^2 \rangle$. (a) For what values of a, b is $\mathbf{F}$ conservative? (b) Using these values, find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$. (c) Using these values, give the equation of a surface S with the property that

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = 0$$

for any two points P, Q on the surface S.

14. A surface S is parameterized by

$$r(u, v) = \langle u, v, uv \rangle \qquad u^2 + v^2 \leq 1.$$

(a) Find its surface area. (b) Parameterize the boundary curve C of S, oriented positively with respect to the orientation of S given by $r_u \times r_v$.

15. Let S be the graph of the function $f(x, y) = 2 - x^2 - y^2$ which lies above the disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$ in the xy−plane. The surface S is oriented so that the normal vector points upwards. Compute the flux

$$\int\int_S \mathbf{F} \cdot \mathbf{n} dS$$

of the vector field

$$\mathbf{F} = \langle -4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2} \rangle$$

through S using the divergence theorem.

16. A broken wine bottle is placed on the xy-plane as shown in the picture. It consists of a portion of a cylinder of radius 1 centered along the $z-$axis, and its bottom is a unit disk in the $xy-$plane centered at the origin. Let C be the path along the broken edge oriented as shown in the picture, and let $\mathbf{F} = \langle -y, 2x, 10z \rangle$. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$.