# Math 53 Spring 2018 Practice Midterm 2

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80 minutes, closed book, closed notes

1. Calculate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)^{2018} dxdy$$

*Solution.* Since the type 2 region $D = \{0 \leq y \leq 1, 0 \leq x \leq \sqrt{1-y^2}\}$ is a quarter circle (by examining the boundary $x = \sqrt{1-y^2}$), and the integrand is a function of $x^2+y^2$, it may be easier to write this in polar coordinates. $D$ may be written as $\{0 \leq \theta \leq \pi/2, 0 \leq r \leq 1\}$ and we have $x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$, so the above integral is equal to

$$\int\int_D (r^2)^{2018} dA = \int_0^{\pi/2}\int_0^1 r^{4036} r dr d\theta = \int_0^{\pi/2} \frac{1^{4038}}{4038} d\theta = \frac{\pi}{2 \cdot 4038}.$$

2. Find the volume of the region enclosed by the plane $z = 4$ and the surface $z = (2x-y)^2 + (x+y-1)^2$ (hint: change of variables)

*Solution.* See https://math.berkeley.edu/ auroux/53f17/prac2Bsol.pdf problem 5.

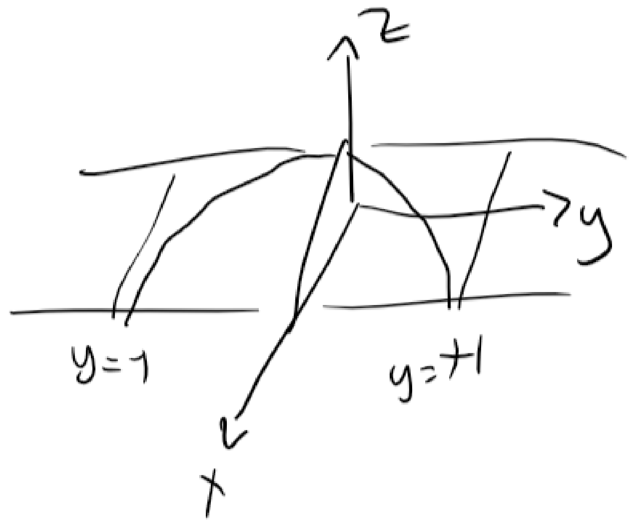
3. Suppose $E$ is the region in space bounded by the surface

$$z = 1 - y^2$$

and the planes $x + z = 1, x = 0$, and $z = 0$. (a) Set up (but do not evaluate) an iterated integral for the mass of a solid with shape $E$ and density $\rho(x, y, z) = xyz$ in the order $dxdydz$. (b) Set up (but do not evaluate) an integral for the same quantity but in the order $dzdxdy$.

*Solution.* The reason this question is tricky is that the region is described as being bounded by surfaces, and not directly as inequalities, so the first step is to get a description in terms of inequalities.

To do this, it is helpful to draw a picture. Each surface by itself is easy: the three planes are the boundary of an (infinite) triangular tent with vertical back wall, horizontal floor, and top which slopes down at a 45 degree angle. $z = 1-y^2$ is an upside down parabolic cylinder, so you can think of it as a curved tent. The region is the intersection of these




two tents, pictured below.

We now see that the region can be described by the *inequalities*

$$x \geq 0,$$

$$z \geq 0,$$

$$x + z \leq 1,$$

$$z \leq 1 - y^2.$$

(a) To find the limits we ask: (1) what is the range of $z$? We have $0 \leq z \leq 1$ since the second and third inequalities imply $z \leq 1$. (2) what is the range of $y$ given $z$? The only inequality relating the two is the third one, so $-\sqrt{1-z} \leq y \leq \sqrt{1-z}$. (3) what is the range of $x$ given $y$ and $z$? The third inequality is irrelevant here, so we just get $0 \leq x \leq 1 - z$. Thus the integral is[1]:

$$M = \int\int\int_E xyz dV = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_0^{1-z} xyz dx dy dz.$$

(b) This is a bit trickier. The only inequality involving $y$ is the last one, so setting $z = 0$ we see that the range of $y$ is $[-1, 1]$. To find the range of $x$ given $y$, we observe that when $z = 0$ we can achieve any $0 \leq x \leq 1$ (this corresponds to the floor of the tent being rectangular), and it is never possible to get any $x > 1$ by the first and third inequalities. The range of $z$ given $x$ and $y$ is the trickiest. Algebraically, we have two upper bounds on $z$: $z \leq 1 - x$ and $z \leq 1 - y^2$, and at any given $(x, y)$ the one which is smaller will apply. To find which one it is, we calculate the points at which they are the same, namely

$$1 - x = 1 - y^2 \to x = y^2,$$

[1] An earlier version of this solution had a typo where it said $0 \leq x \leq 1 - z^2$

which is also the curve of intersection of the surfaces $x + z = 1$ and $z = 1 - y^2$. Thus, for $x \in [0, y^2]$ the second bound is smaller, and for $x \in [y^2, 1]$ the first bound is smaller. Geometrically, this corresponds to saying: when $x \le y^2$ the ceiling of the tent is curved, and when $x \ge y^2$ it is flat. Combining these facts, we have:

$$M = \int\int\int_E xyzdV = \int_{-1}^{1}\int_{0}^{y^2}\int_{0}^{1-y^2} xyzdzdxdy + \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} xyzdzdxdy.$$

4. Find the volume of the part of the ball $\rho \le a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$, where all regions are given in spherical coordinates.

   *Solution.* Problem 27, Section 15.8 in the textbook.

5. Consider the force field $\mathbf{F} =< x, x >$. Let $C$ be the curve consisting of a straight line segment from $P = (-1, 0)$ to $Q = (0, 1)$ with unit speed, and then another from $Q = (0, 1)$ to $R = (1, 0)$ with unit speed. (a) Find the work done by $\mathbf{F}$ on a particle moving along $C$. (b) Find a curve connecting $P$ and $R$ along which the work done is zero.

   *Solution.* (a) We first parameterize the two parts of the curve $C$. Since line integrals are independent of parameterization, we may use any parameterized curve moving in a straight line from $P$ to $Q$ and then $Q$ to $R$ (i.e., the unit speed specification is irrelevant); for simplicity, let us use

$$C_1 : \mathbf{r}(t) = \langle -1, 0\rangle + t\langle 1, 1\rangle = \langle -1 + t, t\rangle, \quad t \in [0, 1]$$

$$C_2 : \mathbf{r}(t) = \langle 0, 1\rangle + t\langle 1, -1\rangle = \langle t, 1 - t\rangle, \quad t \in [0, 1].$$

The work $\int_C \mathbf{F} \cdot d\mathbf{r}$ is then the sum of

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle -1 + t, -1 + t\rangle \cdot \langle 1, 1\rangle dt = \int_0^1 (-2 + 2t)dt = -2 + 1 = -1$$

and

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t, t\rangle \cdot \langle 1, -1\rangle dt = \int_0^1 0dt = 0,$$

which is $-1$.

(b) Let $\mathbf{r}(t) =< x(t), y(t) >= \langle -1, 0\rangle + t\langle 2, 0\rangle, t \in [0, 1]$ be the line segment joining $P$ and $R$. The work done along this curve is:

$$\int_a^b \langle x(t), x(t)\rangle \cdot \mathbf{r}'(t)dt = \int_0^1 \langle -1+2t, -1+2t\rangle \cdot \langle 2, 0\rangle dt = 2\int_0^1 -1+2tdt = -1+1-0 = 0.$$

This may seem like magic, but the way to come up with this curve is to observe that the vector field depends linearly on $x$, and the two endpoints $P$ and $R$ are symmetric in the $x-$direction, so the work should exactly cancel out along the straight path between them.

6. Find the values of $a$ and $b$ for which the vector field

$$\mathbf{F} =< axy^2 + x^2, x^2y + bx >,$$

is conservative. For these $a$ and $b$, find a potential $f$ such that $\mathbf{F} = \nabla f$.

*Solution.* Recall that a vector field on $\mathbb{R}^2$ is conservative iff $Q_x = P_y$ everywhere. In our case, this happens when

$$2xy + b = 2axy,$$

which happens precisely when $a = 1$ and $b = 0$.

To find a potential $F = \nabla f$ in this case, we must solve the equations

$$xy^2 + x^2 = f_x$$

$$x^2y = f_y.$$

Integrating the first equation, we obtain

$$f(x, y) = x^2y^2/2 + x^3/3 + g(y)$$

for some function $g$. Substituting into the second, we have:

$$x^2y = x^2y + g'(y),$$

so $g'(y) = 0$ which means $g(y) = C$ for some constant. Thus we must have

$$f(x, y) = x^2y^2/2 + x^3/3 + C.$$

7. True or False:

(a) The line integral of a vector field $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the direction of the curve $C$ (i.e., does not change if the direction is reversed).

(b) If $F$ is a vector field such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ and $\int_C \mathbf{F} \cdot \mathbf{n}ds = 0$ for every simple closed curve $C$ in the plane, then $\mathbf{F}$ must be constant.

(c) If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for *every* curve $C$ in the plane, then $\mathbf{F}$ must be identically zero.

*Solution.* (a) False. Reversing the orientation of a curve negates the line integral, a fact which we have used multiple times (e.g., in establishing independence of path, and in the proof of Green's theorem). (b) False. By Green's theorem, any vector field satisfying $curl(F) = Q_x - P_y = 0$ and $div(F) = P_x + Q_y = 0$ has this property. The first condition is satisfied by any conservative field $F = \nabla f$. The second condition then becomes $f_{xx} + f_{yy} = 0$, which is satisfied by many $f$ such as $f(x, y) = x^2 - y^2$, for which we have $\nabla f = \langle 2x, -2y \rangle$, which is not constant. (c) True. First, if the condition is true for every *closed* curve then we must have $\mathbf{F} = \nabla f$ for some $f$. The fundamental theoreom for line integrals along with the assumed condition then implies that for every pair of points $a, b$, we have

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = f(b) - f(a),$$

where $C$ is any curve connecting $a$ and $b$. So $f$ must be constant, which means $\mathbf{F}$ must be identically zero.

8. Consider the parameterized curves:

$$C_1 : \mathbf{r}(t) =< t, \sin(3t) >, t \in [0, \pi]$$

$$C_2 : \mathbf{r}(t) =< \pi/2 + (\pi/2)\cos(t), (\pi/2)\sin(t) >, t \in [0, \pi].$$

Let $C := C_1 + C_2$ (i.e., $C_1$ followed by $C_2$). (a) Draw a sketch of $C$. (b) Use Green's theorem applied to an appropriate vector field to find the area enclosed by $C$. (c) Let $\mathbf{F} =< x + y + \pi, x + y + 2\pi >$. What is the flux $\int_C \mathbf{F} \cdot \mathbf{n} ds$ of $F$ across $C$?

*Solution.* (a) The first curve is just the graph of $\sin(3t)$ in $[0, \pi]$, and the second curve is a semicircle of radius $\pi/2$ centered at $\pi/2$. These curves do not intersect. (b) Let $D$ be the domain contained in the interior of $C$. Taking $P = -y, Q = 0$, Green's theorem states that

$$\int_D (0 - (-1))dA = -\int_C y dx = -\int_{C_1} y(t)x'(t)dt - \int_{C_2} y(t)x'(t)dt.$$

The first integral is

$$\int_0^\pi \sin(3t)(1)dt = (-\cos(3t)/3)|_0^\pi = 1/3 + 1/3 = 2/3,$$

and the second is

$$\int_0^\pi (\pi/2)\sin(t)(\pi/2)(-\sin(t))dt = -(\pi^2/4)\int_0^\pi \sin^2 t dt = -\pi^3/8,$$

so the desired area is

$$\pi^3/8 - 2/3.$$

In retrospect, this question was too hard, because it is not easy to see that the curves $C_1$ and $C_2$ do not intersect. I will not give such subtle questions on the real exam.

(c) By the divergence form of Green's theorem, the flux is $\int\int_D (P_x + Q_y)dA = \int\int_D (1 + 1)dA$ which is twice the area, which is $2(\pi^3/8 - 2/3)$.