Practice Midterm 1 Solutions

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(1) When $\bar{r}(t) = \langle t^3-1,\ t^2+2t,\ \ln(t) - 1/t \rangle$ crosses the yz-plane we have

$$t^3 - 1 = 0 \longrightarrow t = 1 \Rightarrow \bar{r}(t) = \langle 0, 3, -1 \rangle.$$

The velocity at $t=1$ is

$$\bar{r}'(t)\Big|_{t=1} = \left\langle 3t^2, 2t+2, \frac{1}{t} + \frac{1}{t^2} \right\rangle \Big|_{t=1}$$

$$= \langle 3, 4, 2 \rangle.$$

The tangent vector is parallel to the velocity, so the required angle satisfies

$$\cos\theta = \frac{\bar{v} \cdot \bar{r}'(1)}{|\bar{v}|\,|\bar{r}'(1)|}$$

$$= \frac{\langle 1,2,2 \rangle \cdot \langle 3,4,2 \rangle}{\sqrt{1^2+2^2+2^2}\ \sqrt{3^2+4^2+2^2}}$$

$$= \frac{3+8+4}{\sqrt{9}\ \sqrt{29}} = \underline{5/\sqrt{29}}$$

(2) See Problem 3 of https://math.berkeley.edu/~auroux/53f17/prac1Bsol.pdf

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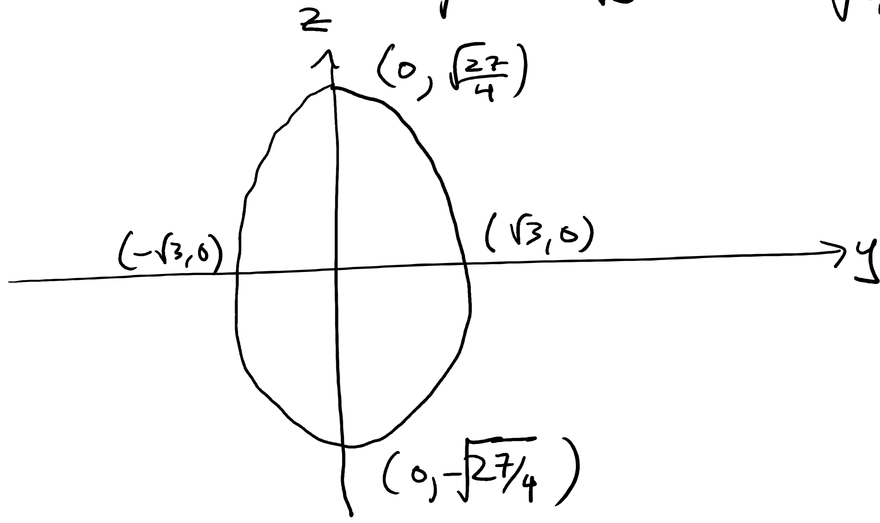
Trace in $x = 1/2$ satisfies: $\left(\frac{1}{2}\right)^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$

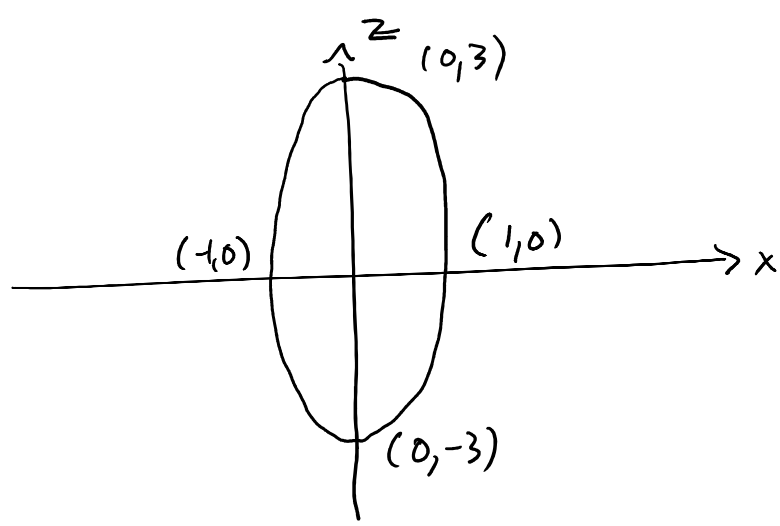
$$\iff \frac{y^2}{4} + \frac{z^2}{9} = 3/4$$

$$\iff \frac{y^2}{3} + \frac{z^2}{27/4} = 1$$

which is an ellipse with axis lengths $\sqrt{3}$ and $\sqrt{\frac{27}{4}} \approx 3$



The $y=0$ trace satisfies $x^2 + \frac{z^2}{9} = 1$, which is an ellipse with sides 1 and 3



(3)(b) The tangent plane to S at any point is perpendicular to the gradient:

$$\nabla f = \langle 2x, y/2, 2z/9 \rangle.$$

So at the point we are looking for, we must have $\nabla f \parallel \langle 1,1,1 \rangle$ i.e.

$$2x = c \cdot 1$$
$$y/2 = c \cdot 1 \qquad \text{for some constant } c.$$
$$2z/9 = c \cdot 1$$

Moreover, since P lies on $f(x,y,z)=1$

$$x^2 + y^2/4 + z^2/9 = 1 \Rightarrow \frac{c^2}{4} + c^2 + \frac{9}{4}c^2 = 1$$
$$\Rightarrow c^2 = \frac{4}{14} = 2/7$$
$$\Rightarrow c = \pm \sqrt{2/7}.$$

So there are two such points,

$$P = \pm \sqrt{\tfrac{2}{7}} \left(\tfrac{1}{2}, 2, \tfrac{9}{2} \right)$$ (either one is a valid answer).

(c) Let's take the plane through $P = \sqrt{\frac{2}{7}}\left(1/2, 2, 9/2 \right)$ perpendicular to $\langle 1,1,1 \rangle$:

$$\boxed{\left(\bar{r} - \sqrt{\tfrac{2}{7}} \langle 1/2, 2, 9/2 \rangle \right) \cdot \langle 1,1,1 \rangle = 0.}$$

Plugging in $\bar{r} = \langle 0,0,0 \rangle$ we have

$$-\sqrt{\tfrac{2}{7}} \langle 1/2, 2, 9/2 \rangle \cdot \langle 1,1,1 \rangle = -\sqrt{\tfrac{2}{7}} \left(\tfrac{1}{2} + 2 + \tfrac{9}{2} \right) \neq 0,$$

So the origin is not on the plane.

(4) Let $\bar{r}(t)$ be a differentiable parameterized curve moving with constant speed along the contour C containing $(0,0)$. Suppose $\bar{r}(t_1) = \bar{r}(t_2) = (0,0)$ for two times $t_1 \neq t_2$, and assume $\bar{r}(t)$ doesn't stop or end at t_1 or t_2 so $\bar{r}'(t_1)$ and $\bar{r}'(t_2)$ are well defined.

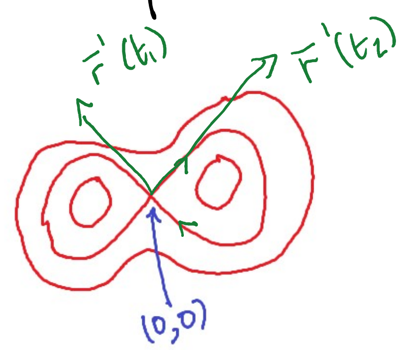
We know that

$$\frac{d}{dt} f(r(t)) \Big|_{t_1} = \nabla f(0,0) \cdot \bar{r}'(t_1) = 0$$

$$\frac{d}{dt} f(r(t)) \Big|_{t_2} = \nabla f(0,0) \cdot \bar{r}'(t_2) = 0.$$

However, $\bar{r}'(t_1)$ and $\bar{r}'(t_2)$ are not parallel from the picture, so $\nabla f(0,0)$ cannot be perpendicular to both of them.

Thus we must have $\nabla f(0,0) = 0$, so $(0,0)$ is a critical point of f. $\bar{r}'(t_2)$ $\bar{r}'(t_1)$



(5) C consists of points satisfying $x = \pm\sqrt{2-y^2}$
$y = z$

So we may parameterize the half of it containg $(1,1,1)$ as:

$z(t) = t$
$y(t) = t$
$x(t) = \sqrt{2-t^2}$

or $\bar{r}(t) = \langle \sqrt{2-t^2}, t, t \rangle$,

for which we have $\bar{r}(1) = \langle 1, 1, 1 \rangle$.

The velocity at $t=1$ is:

$$\bar{r}'(t)\Big|_{t=1} = \left\langle \frac{-2t}{2\sqrt{2-t^2}}, 1, 1 \right\rangle\Big|_{t=1}$$

$$= \langle -1, 1, 1 \rangle.$$

So the tangent line at at $(1,1,1)$ is given by

$$\bar{s}(t) = \langle 1,1,1 \rangle + t\langle -1,1,1 \rangle$$

used a different variable to avoid confusion with $\bar{r}(t)$.

$$= \underline{\underline{\langle 1-t, 1+t, 1+t \rangle}}.$$

6. $z = xe^{-y}$, $x = \sin(s+t)$, $y = \cos(s-t)$

has the dependency diagram

$$\begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] \arrow[drr, dash] & & y \arrow[d, dash] \arrow[dll, dash] \\ s & & t \end{tikzcd}$$

By the chain Rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= e^{-y}\cdot\cos(s+t) + (-xe^{-y})(-\sin(s-t))$$

$$= e^{-y}\left(\cos(s+t) + x\sin(s-t)\right)$$

$$= e^{-\cos(s-t)}\left(\cos(s+t) + \sin(s+t)\sin(s-t)\right).$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= e^{-y}\left(\cos(s+t)\right) + (-xe^{-y})\left(-\sin(s-t)(-1)\right)$$

$$= e^{-y}\left(\cos(s+t) - x\sin(s-t)\right)$$

$$= e^{-\cos(s-t)}\left(\cos(s+t) - \sin(s+t)\sin(s-t)\right).$$

7 The total differential is given by:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \frac{2x}{2\sqrt{x^2+4y^2+z^2}}dx + \frac{8y}{2\sqrt{x^2+4y^2+z^2}}dy + \frac{2z}{2\sqrt{x^2+4y^2+z^2}}dz.$$

At $(3,1,6)$ we have $\sqrt{3^2+4\cdot 1+6^2} = \sqrt{49} = 7$, so

at $(3,1,6)$: $df = \frac{3}{7}dx + \frac{4}{7}dy + \frac{6}{7}dz$.

This means that the linear approximation to f at $(3,1,6)$ is:

$$f(x,y,z) - f(3,1,6) \approx \frac{3}{7}(x-3) + \frac{4}{7}(y-1) + \frac{6}{7}(z-6)$$

Plugging in $(x,y,z) = (3.02, 0.99, 5.97)$ we have

$$f(3.02, 0.99, 5.97) \approx f(3,1,6) + \frac{3}{7}(.02) + \frac{4}{7}(-.01) + \frac{6}{7}(-.03)$$

$$= 7 - \frac{.16}{7}$$ (no need to simplify further)

To compute the directional derivative, we observe that
$\nabla f(3,1,6) = \langle 3/7, 4/7, 6/7 \rangle$, so

$$D_{\vec{v}} f(3,1,6) = \nabla f(3,1,6) \cdot \langle 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3} \rangle$$

$$= 1/\sqrt{3}\left(3/7 - 4/7 + 6/7\right) = \underline{\underline{5/7\sqrt{3}}}$$

(8) We compute the partial derivatives:

$$f_x = 4x^3 - 4y = 0 \quad \text{at a critical pt}$$
$$f_y = 4y - 4x = 0$$
$$\Downarrow$$
$$y = x \quad \text{and} \quad 4x^3 - 4x = 0$$
$$\parallel$$
$$4x(x^2-1) = 0$$

So there are <u>three</u> critical points: $(0,0)$, $(1,1)$, $(-1,-1)$.

We now compute the second derivatives and apply the second derivative test:

	Formula	$(0,0)$	$(1,1)$	$(-1,-1)$
f_{xx}	$12x^2$	0	12	12
f_{yy}	4	4	4	4
$f_{xy} = f_{yx}$	-4	-4	-4	-4
D	$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$	$\begin{vmatrix} 0 & -4 \\ -4 & 4 \end{vmatrix} = -16$	$\begin{vmatrix} 12 & -4 \\ -4 & 4 \end{vmatrix} = 48 - 16 = 32$	$\begin{vmatrix} 12 & -4 \\ -4 & 4 \end{vmatrix} = 32$
		$\Downarrow$ Saddle pt	$\Downarrow$ local min since $f_{xx} > 0$	$\Downarrow$ local min since $f_{xx} > 0$

(9) The closest pt to $(0,0,0)$ on the plane also minimizes the squared distance $(x-0)^2+(y-0)^2+(z-0)^2$, so we can write this as an optimization problem:

minimize $f(x,y,z) = x^2+y^2+z^2$

subject to $g(x,y,z) = 2x+y-z = 6$

The partial derivatives are: $f_x = 2x \quad f_y = 2y \quad f_z = 2z$

$g_x = 2 \quad g_y = 1 \quad g_z = -1$

So the Lagrange Multiplier equations give:

$$2x = 2\lambda \implies x = \lambda$$
$$2y = \lambda \implies y = \lambda/2$$
$$2z = -\lambda \implies z = -\lambda/2.$$

We further have $2x+y-z = 2\lambda + \lambda/2 + \lambda/2 = 6$

So $\underline{\underline{\lambda = 2}}$.

Thus, the closest point on the plane is

$\underline{\underline{(2, 1, -1)}}$.

(10) See Problem 10 of https://math.berkeley.edu/~auroux/53f17/prac1Bsol.pdf