Math 53 First Midterm Exam, Prof. Srivastava
February 22, 2018, 5:10pm–6:30pm, 155 Dwinelle Hall.

Name: ____________________________________________________________

SID: __________________________________________________________________

GSI: __________________________________________________________________

Name of the student to your left: ______________________________________
Name of the student to your right: ______________________________________

Instructions: Write all answers clearly in the provided space. This exam includes some space for scratch work at the bottom of pages 2 and 4 which will not be graded. Do not under any circumstances un staple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious and will not be given full credit. Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Sign here: _________________________________________________________

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Do not turn over this page until your instructor tells you to do so.
1. (10 points) Find a parametric equation of the line defined by the intersection of the planes:

\[ x - y + 3z = 5 \]
\[ 3x + y - z = 3. \]
2. The top extremity of a ladder of unit length rests against a vertical wall, while the bottom is being pulled away, as shown below.

(a) (8 points) Find a parameterized curve $\mathbf{r}(\theta)$ tracing the trajectory of the midpoint $P$ of the ladder as it goes from fully vertical to horizontal, using as parameter the angle $\theta$ between the ladder and the vertical wall, and treating the point at which the wall and the ground meet as the origin.

(b) (2 points) Is the speed of $P$ (as a function of $\theta$) increasing, decreasing, or constant as $\theta$ varies from 0 to $\pi/2$?
3. Consider the function \( f(x, y) = xy \).

   (a) (5 points) Roughly sketch the level curves \( f(x, y) = 1 \), \( f(x, y) = 2 \), and \( f(x, y) = -1 \), labeling which is which.

   (b) (5 points) Find a point \( P \) on the curve \( f(x, y) = 1 \) at which the directional derivative along the direction \( \mathbf{u} = \frac{1}{\sqrt{2}} \mathbf{i} - \frac{1}{\sqrt{2}} \mathbf{j} \) is equal to zero.

   (c) (3 points) Sketch the direction of the vector \( \nabla f \) at the point \( P \) that you found above, originating from \( P \) in the drawing in part (a).
4. (10 points) The parameterized curves $\mathbf{a}(t) = \langle 3t, t^2 - 2, e^t + 1 \rangle$ and $\mathbf{b}(s) = \langle s^2 - s, -2s, s^3 + s^2 \rangle$ intersect at the point $P = (0, -2, 2)$. Find the cosine of the (acute) angle of their intersection at $P$. 

[Scratch Space Below]
5. Consider the function $f(x, y) = \sqrt{e^x + 3e^y - 1}$.

(a) (5 points) Find the total differential of $f$ at $x = 0, y = 1$.

(b) (5 points) Find the equation of the tangent plane to the graph $\{(x, y, z) : z = f(x, y)\}$ of $f$ at the point $P = (0, 1, f(0, 1))$. 
(c) (5 points) Use the equation from (b) to compute an approximation to the value of $f(.01, .99)$.

(d) (3 points) Is there a point $Q$ on the graph at which the tangent plane to the graph is horizontal (i.e., parallel to the $xy$-plane)? If so, find such a point, and if not explain why.

(e) (3 points) Is there a point $R$ on the graph where the tangent plane is vertical (i.e., perpendicular to the $xy$-plane)? If so, find such a point, and if not explain why.
6. (10 points) Let \( w = w(x, y) \) be a differentiable function and let \( x = s^2 t \) and \( y = s + 1/t \). Use the chain rule to express the partial derivatives \( w_s \) and \( w_t \) in terms of \( w_x, w_y, s, \) and \( t \).

7. (10 points) Suppose \( z(x, y) \) is a differentiable function satisfying the equation

\[
x^2 - y^2 + z^2 - 2z = 4.
\]

Find \( \partial z/\partial x \) and \( \partial^2 z/\partial x^2 \) in terms of \( x, y, z \).
8. Let $A(x, y)$ be the area of the triangle with vertices at the points $P = (0, 0), Q = (1, 2)$, and $R = (x, y)$.
   
   (a) (5 points) Derive a formula for $A(x, y)$.

   (b) (5 points) Identify all critical points of $A(x, y)$, and classify them as local minima, maxima, or saddle points, with justification.
(c) (6 points) Use Lagrange multipliers to find a point $R = (x, y)$ on the curve

$$xy = -1$$

which minimizes $A(x, y)$, and determine the minimum area. (hint: it might be easier to minimize $A^2$)