Math 53 Homework 7 Solutions

Section 15.4

4. To find the mass of the lamina, we integrate $\rho(x, y)$ over the box:

$$\begin{aligned}
m &= \int_0^a \int_0^b 1 + x^2 + y^2 \, dy \, dx \\
&= \int_0^a y + x^2 y + \frac{y^3}{3} \Big|_{y=0}^{y=b} dx \\
&= \int_0^a b + bx^2 + \frac{b^3}{3} \, dx \\
&= bx + \frac{bx^3}{3} + \frac{b^3 x}{3} \Big|_{x=0}^{x=a} \\
&= ab + \frac{a^3 b}{3} + \frac{ab^3}{3} \\
&= \frac{a^3 b + 3ab + ab^3}{3}.
\end{aligned}$$

We then integrate $x\rho(x, y)$, which gives

$$\begin{aligned}
\int_0^a \int_0^b x + x^3 + xy^2 \, dy \, dx &= \int_0^a xy + x^3 y + \frac{xy^3}{3} \Big|_{y=0}^{y=b} dx \\
&= \int_0^a bx + bx^3 + \frac{b^3 x}{3} \, dx \\
&= \frac{bx^2}{2} + \frac{bx^4}{4} + \frac{b^3 x^2}{6} \Big|_{x=0}^{x=a} \\
&= \frac{a^2 b}{2} + \frac{a^4 b}{4} + \frac{a^2 b^3}{6} \\
&= \frac{3a^4 b + 6a^2 b + 2a^2 b^3}{12}.
\end{aligned}$$

Dividing by the mass of the lamina, we have

$$\overline{x} = \frac{3a^4 b + 6a^2 b + 2a^2 b^3}{4(a^3 b + 3ab + ab^3)}.$$

Similarly,

$$\overline{y} = \frac{2a^3 b^2 + 6ab^2 + 3ab^4}{4(a^3 b + 3ab + ab^3)}.$$

Therefore, the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{3a^4 b + 6a^2 b + 2a^2 b^3}{4(a^3 b + 3ab + ab^3)}, \frac{2a^3 b^2 + 6ab^2 + 3ab^4}{4(a^3 b + 3ab + ab^3)} \right).$$

8. In the region bounded by $y = x + 2$ and $y = x^2$, x ranges from -1 to 2 (the two roots of $x^2 = x + 2$). Thus, the mass is

$$\begin{aligned} m &= \int_{-1}^{2} \int_{x^2}^{x+2} kx^2 \, dy \, dx \\ &= \int_{-1}^{2} kx^2 y \Big|_{y=x^2}^{y=x+2} \, dx \\ &= \int_{-1}^{2} kx^2(x + 2 - x^2) \, dx \\ &= \frac{kx^4}{4} + \frac{2kx^3}{3} - \frac{kx^5}{5} \Big|_{x=-1}^{x=2} \\ &= \frac{63k}{20}. \end{aligned}$$

We now integrate $x\rho(x, y)$:

$$\begin{aligned} \int_{-1}^{2} \int_{x^2}^{x+2} kx^3 \, dy \, dx &= \int_{-1}^{2} kx^3 y \Big|_{y=x^2}^{y=x+2} \, dx \\ &= \int_{-1}^{2} kx^3(x + 2 - x^2) \, dx \\ &= \frac{kx^5}{5} + \frac{kx^4}{2} - \frac{kx^6}{6} \Big|_{x=-1}^{x=2} \\ &= \frac{64k}{15}. \end{aligned}$$

Integrating $y\rho(x, y)$ gives

$$\begin{aligned} \int_{-1}^{2} \int_{x^2}^{x+2} kx^2 y \, dy \, dx &= \int_{-1}^{2} \frac{kx^3y^2}{2} \Big|_{y=x^2}^{y=x+2} \, dx \\ &= \frac{1}{2} \int_{-1}^{2} kx^3(x + 2 - x^2)^2 \, dx \\ &= \frac{k}{2} \int_{-1}^{2} x^3(x^4 - 2x^3 - 3x^2 + 4x + 4) \, dx \\ &= \frac{k}{2} \left(\frac{x^8}{8} - \frac{2x^7}{7} - \frac{x^6}{2} + \frac{4x^5}{5} + x^4 \Big|_{x=-1}^{x=2} \right) \\ &= \frac{1377k}{560}. \end{aligned}$$

After dividing by the mass, we get

$$(\overline{x}, \overline{y}) = \left(\frac{256}{189}, \frac{153}{196} \right).$$

12. Since the density is proportional to the square of the distance from the origin, we have $\rho(x, y) = k(x^2 + y^2)$ for some constant k. In terms of polar

coordinates, this is just $\rho(r,\theta) = kr^2$. The mass is therefore

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^1 kr^2 \cdot r \, dr \, d\theta \\ &= 2\pi \cdot \frac{k}{4} \\ &= \frac{k\pi}{2}. \end{aligned}$$

To integrate $x\rho(x,y)$, we just multiply the integrand by $r\cos\theta$, so we get

$$\int_0^{2\pi} \int_0^1 kr^3 \cos\theta \cdot r \, dr \, d\theta.$$

This is just 0 since the integral of $\cos\theta$ from 0 to 2π is 0. Since the same holds for $\sin\theta$, the integral of $y\rho(x,y)$ is also 0, so the center of mass is $(0,0)$.

Section 15.9

2. The Jacobian is

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \det\begin{pmatrix} 2u+v & u \\ v^2 & 2uv \end{pmatrix} \\ &= (2u+v)\cdot 2uv - uv^2 \\ &= 4u^2v + 2uv^2 - uv^2 \\ &= 4u^2v + uv^2. \end{aligned}$$

4. The Jacobian is

$$\begin{aligned} \frac{\partial(x,y)}{\partial(p,q)} &= \det\begin{pmatrix} e^q & pe^q \\ qe^p & e^p \end{pmatrix} \\ &= e^{q+p} - pqe^{q+p} \\ &= (1-pq)e^{p+q}. \end{aligned}$$

8. Since $x = v$ and v ranges from 0 to 1, x can take on any value from 0 to 1. For fixed x, $y = u(1+x^2)$, so as u varies from 0 to 1, y varies from 0 to $1+x^2$. Thus, the image of the transformation is the region bounded by $x = 0, x = 1, y = 0$, and $y = 1+x^2$.

12. We may as well try to find a linear transformation sending the unit square to the given parallelogram. Say

$$T(u,v) = (au+bv, cu+dv)$$

for some constants a,b,c,d. We want T to satisfy

$$\begin{aligned} T(1,0) &= (4,3), \\ T(0,1) &= (-2,1), \end{aligned}$$

since then $T(0,0)$ and $T(1,1)$ will automatically be sent to the other two points. These two constraints give $a = 4, b = -2, c = 3, d = 1$, so the change of variables is

$$\begin{aligned} x &= 4u - 2v, \\ y &= 3u + v. \end{aligned}$$

16. Inverting the change of variables gives

$$\begin{aligned} u &= x - y, \\ v &= 3x + y. \end{aligned}$$

Substituting in the four vertices of the paralellogram, we get the points $(4,0), (-4,0), (4,8), (-4,8)$ in the uv-plane. Since the transformation is linear, the image in the uv-plane will be a rectangle with these four vertices. The Jacobian is just

$$\det \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{pmatrix} = \frac{1}{4},$$

so we have

$$\begin{aligned} \iint_R 4x + 8y \, dA &= \frac{1}{4} \int_{-4}^{4} \int_0^8 4 \cdot \frac{1}{4}(u+v) + 8 \cdot \frac{1}{4}(v - 3u) \, dv \, du \\ &= \frac{1}{4} \int_{-4}^{4} \int_0^8 -5u + 3v \, dv \, du \\ &= \frac{1}{4} \int_{-4}^{4} -5uv + \frac{3v^2}{2} \Big|_{v=0}^{v=8} du \\ &= \int_{-4}^{4} -10u + 24 \, du \\ &= -5u^2 + 24u \Big|_{u=-4}^{u=4} \\ &= 192. \end{aligned}$$

19. The two lines $y = x$ and $y = 3x$ become $u = v^2$ and $u = v^2/3$ in the uv-plane. The two hyperbolas $xy = 1$ and $xy = 3$ are just $u = 1$ and $u = v$. Thus, the new region is the one bounded by $u = 1, u = 3, u = v^2$, and $u = v^2/3$. The Jacobian is

$$\det \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{pmatrix} = \frac{1}{v}.$$

This gives

$$\begin{aligned} \iint_R xy \, dA &= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u \cdot \frac{1}{v} \, dv \, du \\ &= \int_1^3 u \, \log v \Big|_{v=\sqrt{u}}^{v=\sqrt{3u}} du. \end{aligned}$$

But

$$\log(\sqrt{3u}) - \log(\sqrt{u}) = \log(\sqrt{3}),$$

so

$$\iint_R xy\,dA = \int_1^3 u\cdot\log(\sqrt{3})\,du$$
$$= \frac{u^2\cdot\log(\sqrt{3})}{2}\Big|_{u=1}^{u=3}$$
$$= 2\log 3.$$

Section 15.6

6. The integrand does not depend on x, so

$$\int_0^1\int_0^1\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy = \int_0^1\int_0^1 \frac{z\sqrt{1-z^2}}{y+1}\,dz\,dy.$$

We can pull the $1/(y+1)$ out of the integrand, which allows us to write

$$\int_0^1\int_0^1 \frac{z\sqrt{1-z^2}}{y+1}\,dz\,dy = \left(\int_0^1 \frac{1}{y+1}\,dy\right)\left(\int_0^1 z\sqrt{1-z^2}\,dz\right).$$

The first integral is just $\log 2$, and the second is $\frac{1}{3}$ (make the substitution $u = 1-z^2$). Thus, the value of the original integral is $\dfrac{\log 2}{3}$.

8. We have

$$\int_0^1\int_0^1\int_0^{2-x^2-y^2} xye^z\,dz\,dy\,dx = \int_0^1\int_0^1 xy(e^{2-x^2-y^2}-1)\,dy\,dx.$$

We can split this up into two integrals and then factor each into a product:

$$\int_0^1\int_0^1 xy(e^{2-x^2-y^2}-1)\,dy\,dx = \int_0^1\int_0^1 xye^{2-x^2-y^2}\,dy\,dx - \int_0^1\int_0^1 xy\,dy\,dx$$
$$= e^2\left(\int_0^1 xe^{-x^2}\,dx\right)\left(\int_0^1 ye^{-y^2}\,dy\right) - \left(\int_0^1 x\,dx\right)\left(\int_0^1 y\,dy\right).$$

To integrate xe^{-x^2}, we substitute $u = x^2$, and it reduces to integrating $\frac{1}{2}e^{-u}$ from 0 to 1. This integral is $\frac{1}{2}\left(1-\frac{1}{e}\right)$, so the original integral is

$$e^2\cdot\left(\frac{1}{2}\left(1-\frac{1}{e}\right)\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{(e-1)^2}{4} - \frac{1}{4}$$
$$= \frac{e^2-2e}{4}.$$

10. Since the values of x depend on y and the values of z depend on x and y, we integrate z, then x, then y. This gives

$$\iiint_E e^{z/y}\,dV = \int_0^1\int_y^1\int_0^{xy} e^{z/y}\,dz\,dx\,dy$$
$$= \int_0^1\int_y^1 ye^{z/y}\Big|_{z=0}^{z=xy}\,dx\,dy$$

$$
\begin{aligned}
&= \int_0^1 \int_y^1 y(e^x - 1)\, dx\, dy \\
&= \int_0^1 y(e^x - x)\Big|_{x=y}^{x=1} dy \\
&= \int_0^1 y[(e-1) - (e^y - y)]\, dy \\
&= \frac{(e-1)y^2}{2} + \frac{y^3}{3} - (y-1)e^y\Big|_{y=0}^{y=1} \\
&= \frac{e-1}{2} + \frac{1}{3} - 1 \\
&= \frac{e}{2} - \frac{7}{6}.
\end{aligned}
$$

14. First notice that y ranges from 0 to 2, independently of x and z, so we should put dy on the outside of the integral. The surfaces $z = x^2 - 1$ and $z = 1 - x^2$ intersect when $x = \pm 1$ and $z = 0$, so x ranges from -1 to 1. This gives the iterated integral

$$
\begin{aligned}
\iiint_E x - y\, dV &= \int_0^2 \int_{-1}^1 \int_{x^2-1}^{1-x^2} x - y\, dz\, dx\, dy \\
&= \int_0^2 \int_{-1}^1 (x-y)\cdot 2(1-x^2)\, dx\, dy \\
&= \int_0^2 \int_{-1}^1 2x - 2x^3 - 2y + 2x^2y\, dx\, dy \\
&= \int_0^2 x^2 - \frac{x^4}{2} - 2xy + \frac{2x^3y}{3}\Big|_{x=-1}^{x=1} dy \\
&= \int_0^2 -\frac{8}{3}y\, dy \\
&= -\frac{4}{3}y^2\Big|_{y=0}^{y=2} \\
&= -\frac{16}{3}.
\end{aligned}
$$

22. Since the only condition on x is that $x^2 + z^2 \le 4$ (so $-2 \le x \le 2$) we should have the outermost integral be with respect to x. Then z varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$, and finally, y varies from -1 to $4 - z$. This gives the iterated integral

$$
\begin{aligned}
\iiint_E dV &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} dy\, dz\, dx \\
&= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 5 - z\, dz\, dx \\
&= \int_{-2}^2 5z - \frac{z^2}{2}\Big|_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} dx \\
&= \int_{-2}^2 10\sqrt{4-x^2}\, dx.
\end{aligned}
$$

This is just 10 times the area of a semicircle of radius 2, so the volume of the solid is $10 \cdot 2\pi = 20\pi$.

28. See the attached diagram.

30. Since x is independent of y and z, the integral with respect to x can be freely moved around. If the integral in y comes before the integral in z (from left to right), then the bounds are

$$-3 \leq y \leq 3,\ -\sqrt{9-y^2} \leq z \leq \sqrt{9-y^2}.$$

Similarly, if z comes before y, then the bounds are

$$-3 \leq z \leq 3,\ -\sqrt{9-z^2} \leq y \leq \sqrt{9-z^2}.$$

This gives the 6 integrals

$$\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_{-2}^{2} f(x,y,z)\,dx\,dy\,dz,$$
$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-2}^{2} f(x,y,z)\,dx\,dz\,dy,$$
$$\int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dx\,dz,$$
$$\int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dy\,dz\,dx,$$
$$\int_{-3}^{3}\int_{-2}^{2}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dx\,dy,$$
$$\int_{-2}^{2}\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dy\,dx.$$

34. Since the conditions on y and z are written in terms of x, we can swap dy and dz in the given integral while keeping the bounds the same. This gives

$$\int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx.$$

If we instead have the order be $dz\ dx\ dy$, then y ranges from 0 to 1, x ranges from 0 to $1-y$, and z ranges from 0 to $1-x^2$. This gives the integral

$$\int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy.$$

If we use the order $dy\ dx\ dz$, then z ranges from 0 to 1, x ranges from 0 to $\sqrt{1-z}$, and finally y ranges from 0 to $1-x$. Thus, we get the integral

$$\int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz.$$

Using the order $dx\ dy\ dz$, z and y both range from 0 to 1. However, the range for x will depend on where in the unit square (y,z) lies. If we project the curve where $z = 1 - x^2$ and $y = 1 - x$ intersect onto the yz-plane, we get

$$z = 1 - (1 - y)^2 = -y^2 + 2y,$$

or equivalently,

$$y = 1 - \sqrt{1 - z}.$$

If the point (y, z) lies below this curve, then x is limited by the plane $y = 1 - x$, so $x \le 1 - y$. If (y, z) lies above the curve, then x will range from 0 to $\sqrt{1 - z}$. Therefore, we have to write the integral as a sum:

$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x, y, z)\; dx\; dy\; dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x, y, z)\; dx\; dy\; dz.$$

Similarly, if we take the order to be $dx\; dz\; dy$, we get

$$\int_0^1 \int_0^{-y^2+2y} \int_0^{1-y} f(x, y, z)\; dx\; dz\; dy + \int_0^1 \int_{-y^2+2y}^1 \int_0^{\sqrt{1-z}} f(x, y, z)\; dx\; dz\; dy.$$