Math 53 Homework 6

due Wednesday, March 7

Section 15.1

Problem 11.

Proof. The solid is a trapezoid with square base $[0,1] \times [0,1]$ in the xy plane, and edges along $z = 4$ and $z = 2$ in the x direction. The volume of this is

$$2 \times 1 \times 1 + \frac{1}{2}(2 \times 1 \times 1) = 3.$$

We check:

$$\int\int_R (4-2y)\,dA = \int_0^1 \int_0^1 (4-2y)\,dx\,dy = (4y - y^2)\Big|_{y=0}^{y=1} = 3$$

□

Problem 13.

Proof. First, we find

$$\int_0^2 (x + 3x^2y^2)\,dx = (\frac{1}{2}x^2 + x^3y^2)\Big|_0^2 = 2 + 8y^2.$$

Next, we find

$$\int_0^3 (x + 3x^2y^2)\,dy = (xy + x^2y^3)\Big|_0^3 = 3x + 27x^2.$$

□

Problem 24.

Proof. We calculate

$$\begin{aligned}
\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx &= \int_0^1 \left(\frac{1}{3}x(x^2+y^2)^{3/2}\right)\Big|_{y=0}^{y=1} dx = && (1)\\
&= \frac{1}{3}\int_0^1 x\big((x^2+1)^{3/2} - x^3\big)\,dx && (2)\\
&= \frac{1}{3}\big(\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{5}x^5\big)\big|_0^1 && (3)\\
&= \frac{1}{15}(2^{5/2} - 2) && (4)
\end{aligned}$$

□

Problem 29.

Proof. The integrand is a product of a function of x and a function of y, which we integrate separately:

$$\int\int_R \frac{xy^2}{x^2+1}\,dA = \int_0^1 x(x^2+1)^{\ 1}\,dx \times \int_{\ 3}^3 y^2\,dy \tag{5}$$

$$= \frac{1}{2}\ln(x^2+1)\big|_{x=0}^{x=1} \times \frac{1}{3}y^3\big|_{y=\ 3}^{y=3} \tag{6}$$

$$= 9\ln 2 \tag{7}$$

□

Problem 32.

Proof. Integrate with respect to y first:

$$\int_0^1\int_0^1 x(1+xy)^{\ 1}\,dy\,dx = \int_0^1 \ln(1+xy)\big|_{y=0}^{y=1}\,dx \tag{8}$$

$$= \int_0^1 \ln(1+x)\,dx \tag{9}$$

$$= (x+1)\ln(x+1) - (x+1)\Big|_0^1 \tag{10}$$

$$= 2\ln 2 - 1, \tag{11}$$

where we guessed the antiderivative of $\ln(x+1)$ by recalling that $x\ln x - x$ is an antiderivative of $\ln x$. □

Section 15.2

Problem 3.

Proof. We calculate

$$\int_0^1\int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \left(\frac{1}{2}x^2\Big|_0^y\right)e^{y^3}\,dy = \int_0^1 \frac{1}{6}(3y^2)e^{y^3}\,dy \tag{12}$$

$$= \frac{1}{6}e^{y^3}\big|_0^1 \tag{13}$$

$$= \frac{e-1}{6} \tag{14}$$

□

Problem 15.

Proof. The intersection points of $y = x - 2$ and $x = y^2$ are $(1,-1)$ and $(4,2)$. The two set-ups (see attached sheet for pictures):

$$\int_{\ 1}^2\int_{y^2}^{y+2} y\,dx\,dy, \qquad \text{and}$$

$$\int_0^1\int_{\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4\int_{x\ \ 2}^{\sqrt{x}} y\,dy\,dx.$$

The first looks easier because the integral with respect to x is immediate, and there's only one integral to evaluate. We will compute the integral in this first way:

$$\int_1^2 \int_{y^2}^{y+2} y\,dx\,dy = \int_1^2 y(y+2-y^2)\,dy \tag{15}$$
$$= y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 \Big|_1^2 \tag{16}$$
$$= (4-1) + \frac{1}{3}(8+1) - \frac{1}{4}(16-1) \tag{17}$$
$$= 6 - \frac{15}{4} \tag{18}$$
$$= \frac{11}{3} \tag{19}$$

□

Problem 19.

Proof. We integrate with respect to x first in order to avoid splitting up the integral (see attached sheet for a picture):

$$\iint_D y^2\,dA = \int_1^2 \int_{y\ \ 1}^{7\ \ 3y} y^2\,dx\,dy = \int_1^2 y^2(8-4y)\,dy \tag{20}$$
$$= \frac{8}{3}y^3 - y^4 \Big|_1^2 \tag{21}$$
$$= \frac{8}{3}(8-1) - (16-1) \tag{22}$$
$$= \frac{56}{3} - 15 \tag{23}$$

□

Problem 24.

Proof. To find the volume, we perform a double integral of $z(x,y) = 1 + x^2y^2$ over the region bounded by $x = y^2$ and $x = 4$. It looks easiest to integrate with resect to x first:

$$\text{volume} = \int_2^2 \int_{y^2}^4 (1 + x^2y^2)\,dx\,dy = \int_2^2 \left(x + \frac{1}{3}x^3y^2 \Big|_{y^2}^4 \right) dy \tag{24}$$
$$= \int_2^2 4 + \frac{64}{3}y^2 - y^2 - \frac{1}{3}y^8\,dy \tag{25}$$
$$= 4y + \frac{61}{9}y^3 - \frac{1}{27}y^9 \Big|_2^2 \tag{26}$$
$$= \frac{2336}{27} \tag{27}$$

□

Problem 25.

Proof. The function is equally easily integrable in both variables, and the triangle is a right triangle in the plane (see attached sheet for a picture), so both orders of integration are of the same difficulty. We'll integrate with respect to x first:

$$\begin{aligned} \text{volume} = \int_1^2 \int_1^{7\ \ 3y} xy\,dx\,dy &= \int_1^2 \frac{1}{2}\big((7-3y)^2-1\big)y\,dy && (28)\\ &= \frac{1}{2}\int_1^2 9y^3-42y^2+48y\,dy && (29)\\ &= \frac{1}{2}\left(\frac{9}{4}y^4-14y^3+24y^2\right)\Bigg|_1^2 && (30)\\ &= \frac{1}{2}(\frac{9}{4}(15)-14(7)+72) && (31)\\ &= \frac{31}{8} && (32) \end{aligned}$$

□

Problem 31.

Proof. Equivalently, we are finding the volume under the function $z = y$ over the quarter unit disk where x, y are non-negative:

$$\begin{aligned} \text{volume} = \int_0^1 \int_0^{\sqrt{1\ \ x^2}} y\,dy\,dx &= \int_0^1 \frac{1}{2}y^2\Big|_{y=0}^{y=\sqrt{1\ \ x^2}} dx && (33)\\ &= \int_0^1 \frac{1}{2}(1-x^2)\,dx && (34)\\ &= \frac{1}{3} && (35) \end{aligned}$$

□

Section 15.3

Problem 1.

Proof. Polar coordinates look like a good choice here: if f is a function on the plane in rectangular coordinates, we would be computing the integral

$$\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)r dr\,d\theta,$$

in whichever order is convenient (by Fubini's theorem). □

Problem 2.

Proof. Rectangular coordinates look best, with either variable integrated against first, as far as the bounds are concerned. We'll integrate in y first:

$$\iint_R f(x,y)\,dA = \int_{\ 1}^1 \int_{\ x}^1 f(x,y)\,dy\,dx.$$

□

Problem 4.

Proof. Polar coordinates:

$$\iint_R f(x,y)\,dA = \int_{\frac{}{4}}^{\frac{3}{4}} \int_0^3 f(r\cos\theta, r\sin\theta)\,r dr\,d\theta.$$

□

Problem 7.

Proof. We compute:

$$\iint_D x^2 y\,dA = \int_0^{\pi}\int_0^5 r^4\cos^2\theta\sin\theta dr\,d\theta = -\frac{1}{3}\cos^3\theta\Big|_0^{\pi} \times \frac{1}{5}r^5\Big|_0^5 \tag{36}$$

$$= \frac{2}{3}5^4 \tag{37}$$

□

Problem 10.

Proof. The region R is the washer between two circles centered at 0, with radii a and b.

$$\iint_R \frac{y^2}{x^2+y^2}\,dA = \int_0^{2\pi}\int_a^b \frac{r^2\sin^2\theta}{r^2} r dr\,d\theta = \int_a^b r\,dr \times \int_0^{2\pi}\sin^2\theta\,d\theta \tag{38}$$

$$= \frac{1}{2}(b-a)^2 \times \int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta \tag{39}$$

$$= \frac{1}{4}(b-a)^2\big(\theta - \frac{1}{2}\sin 2\theta\big)\Big|_0^{2\pi} \tag{40}$$

$$= \frac{\pi}{2}(b-a)^2, \tag{41}$$

where we have used the double angle identity for $\sin^2\theta$ in the second equality. □

Problem 12.

Proof. Our integral is

$$\iint_D \cos\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 \cos(r) r dr\,d\theta = 2\pi \times (r\sin r + \cos r)\Big|_0^2 \tag{42}$$

$$= 2\pi(2\sin 2 + \cos 2 - 1) \tag{43}$$

□

Problem 17.

Proof. We convert to polar coordinates. The circle $x^2+y^2=1$ is given in polar coordinates by the equation $r=1$. The circle $(x-1)^2+y^2=1$ is given by the equation

$$1 = (r\cos\theta - 1)^2 + (r\sin\theta)^2 = r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta,$$

which is the equation $0 = r^2 - 2r\cos\theta$; thus the equation of the second circle is $r = 2\cos\theta$. The points of intersection of the two circles are $\theta = \pm\frac{\pi}{3}$, which should be seen from drawing a diagram as in the attached sheet. This implies that we are integrating θ over the interval $[-\frac{\pi}{3}, \frac{\pi}{3}]$. Since the area of a region R in the plane is the same as the double integral over R of the function 1, we have that

$$\text{area}(R) = \iint_R 1\, dA = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2\cos\theta} 1\, r dr\, d\theta \tag{44}$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}(4\cos^2\theta - 1)\, d\theta \tag{45}$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}(2\cos 2\theta + 1)\, d\theta \tag{46}$$

$$= \frac{1}{2}(\sin 2\theta + \theta)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \tag{47}$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{3}, \tag{48}$$

where we have used another double angle identity. □

Problem 21.

Proof. We observe that, over the unit disk, the plane $z = 4 - 2x - y$ lies entirely above it (since over the unit disk, both x and y are at most 1), and so the volume can be written as an integral of z over the entire unit disk. The function we are integrating is $z(r, \theta) = 4 - 2r\cos\theta - r\sin\theta$. Since we are integrating over a disk, we may choose any order of integration by Fubini's theorem. Integrating $\cos\theta$ or $\sin\theta$ over $[0, 2\pi]$ yields zero, so we may choose to integrate with respect to θ first to simplify things:

$$\int_0^1 \int_0^{2\pi} (4 - 2r\cos\theta - r\sin\theta) r d\theta\, dr = \int_0^1 8\pi r dr \tag{49}$$

$$= 4\pi r^2\Big|_0^1 = 4\pi. \tag{50}$$

□

Problem 22.

Proof. Our domain of integration will be the region between the circle of radius 2 and the circle of radius 4 centered at 0 in the xy plane. If we integrate the function $z = \sqrt{16 - x^2 - y^2}$ over this region, we will only pick up the top half of the volume; so we need to multiply this integral by 2. Polar coordinates make integration feasible:

$$\text{volume} = 2\int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\, r dr\, d\theta \tag{51}$$

$$= 4\pi \times \left(-\frac{1}{3}(16 - r^2)^{3/2} \right)\Big|_2^4 \tag{52}$$

$$= \frac{4\pi}{3} 12^{3/2} \tag{53}$$

□

Problem 26.

Proof. The paraboloid $z = 6 - x^2 - y^2$ opens down, and the paraboloid $z = 2x^2 + 2y^2$ opens up. All of their horizontal sections are circles centered at the origin, so the region we will integrate over will be the disk whose boundary is given by the intersection of the paraboloids: setting the equations equal to each other, we see that this is the circle of radius $\sqrt{2}$. Hence the volume trapped between these paraboloids is

$$\text{volume} = \int_0^{2\pi} \int_0^{\sqrt{2}} ((6 - r^2) - 2r^2)\, r dr\, d\theta = 2\pi \times \int_0^{\sqrt{2}} 6r - 3r^3\, dr \tag{54}$$

$$= 2\pi \times (3r^2 - \frac{3}{4} r^4)\Big|_0^{\sqrt{2}} \tag{55}$$

$$= 2\pi(6 - \frac{3}{4}4) = 6\pi \tag{56}$$

□

Problem 28.

Proof. Part (a) we've already done in problem 22, with $r_1 = 2$ and $r_2 = 4$; looking back, the answer should be

$$4\pi \times \left(-\frac{1}{3}(r_2^2 - r^2)^{3/2} \right)\Big|_{r_1}^{r_2} = \frac{4\pi}{3}(r_2^2 - r_1^2)^{3/2}$$

For part (b), by drawing the right triangle whose hypotenuse is the radius of the sphere r_2 and whose adjacent side is the radius of the drill r_1, we see that the height is in fact $h = \sqrt{r_2^2 - r_1^2}$; thus the answer in part (a) is $\frac{4\pi}{3} h^{3/2}$. □

Problem 32.

Proof. Our domain of integration is the region under the graph of $y = \sqrt{2x - x^2}$ over the interval $[0, 2]$ in x. Squaring this equation and completing the square by adding 1, this is the equation for the top half of the circle of radius 1 centered at $(1, 0)$. Thus we see that θ varies from 0 to $\pi/2$, and in polar coordinates, $y^2 = 2x - x^2$ becomes the relation $r = 2\cos\theta$. This allows us to rewrite our integral, and compute:

$$\int_0^2 \int_0^{\sqrt{2x - x^2}} \sqrt{x^2 + y^2}\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\, r dr\, d\theta \tag{57}$$

$$= \int_0^{\frac{\pi}{2}} \frac{8}{3} \cos^3\theta\, d\theta. \tag{58}$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos\theta(1 - \sin^2\theta)\, d\theta \tag{59}$$

$$= \frac{8}{3}(\sin\theta - \frac{1}{3}\sin^3\theta)\Big|_0^{\frac{\pi}{2}} \tag{60}$$

$$= \frac{8}{3}(1 - \frac{1}{3}) \tag{61}$$

$$= \frac{16}{3} \tag{62}$$

where for the third equality we used $1 = \cos^2\theta + \sin^2\theta$ to rewrite $\cos^3\theta = \cos^2\theta\cos\theta$. □