Solutions to Homework 5

14.7

Problem 3. $\nabla f = \langle 3x^2 - 3y, 3y^2 - 3x \rangle$. If we solve $\nabla f = \mathbf{0}$, we can find two solutions $(0,0)$ and $(1,1)$.

- At $(0,0)$, $f_{xx} = 0, f_{yy} = 0, f_{xy} = -3$ and $D = f_{xx}f_{yy} - f_{xy}^2 < 0$, therefore $(0,0)$ is saddle point.
- At $(1,1)$, $f_{xx} = 6, f_{yy} = 6, f_{xy} = -d$ and $D = 27 > 0$, hence $(1,1)$ is a local minimum.

□

Problem 7. $\nabla f = \langle 1-2xy+y^2, -1-x^2+2xy \rangle$. To solve the equation $\nabla f = \mathbf{0}$, we can add the two equations to get $y^2 - x^2 = 0$, therefore $y = \pm x$. If $y = x$, then the solution is $(1,1)$ and $(-1,-1)$. If $y = -x$, there is no solution.

- At $(1,1)$, $f_{xx} = -2, f_{yy} = 2, f_{xy} = 0$ and $D = -4 < 0$, hence $(1,1)$ is saddle point.
- At $(-1,-1)$, $f_{xx} = 2, f_{yy} = -2, f_{xy} = 0$ and $D = -4 < 0$, hence $(-1,-1)$ is a saddle point.

□

Problem 14. $\nabla f = \langle -y \sin x, \cos x \rangle$. $\nabla f = \mathbf{0}$ implies that $\cos x = 0$. Therefore $\sin x = \pm 1$. Since $y \sin x = 0$, y must be 0. So the critical points are $(\frac{\pi}{2} + k\pi, 0)$ for any integer k.

Since $f_{xx} = -y \cos x$, $f_{yy} = 0$, $f_{xy} = \sin x$, D at $(\frac{\pi}{2} + k\pi, 0)$ is -1. Therefore, all the critical points are saddle points. □

Problem 15. $\nabla f = \langle e^x \cos y, -e^x \sin y \rangle$. Since $e^x \neq 0$, $\nabla f = \mathbf{0}$ implies that $\sin y = \cos y = 0$. However there is no such y. So there is no critical point. □

Problem 31. First step: locate the interior critical point. Since $\nabla f = \langle 2x - 2, 2y \rangle$, we have one critical point $(1,0)$, and the value at $(1,0)$ is -1.

Second step: find maximum and minimum on the boundary:

- On the edge from $(0,-2)$ to $(0,2)$, $f = y^2$, hence the maximum is 4 and the minimum is 0.
- On the edge from $(0,2)$ to $(2,0)$, since $x + y = 2$ and $0 \le x \le 2$, $f = x^2 + (2-x)^2 - 2x = 2x^2 - 6x + 4$. f reaches the minimum $-\frac{1}{2}$ when $x = \frac{3}{2}$ and reaches maximum 4 when $x = 0$.
- On the edge from $(0,-2)$ to $(2,0)$, by the same argument in the second case(or symmetry), the maximum is 4 and minimum is $-\frac{1}{2}$

Putting them together we have the maximum is 4 and minimum is -1. □

Problem 34. First step: locate the interior critical point. Since $\nabla f = \langle 2x + y, x + 2y - 6 \rangle$, we have one critical point $(-2,4)$, and the value at $(-2,4)$ is -12.

Second step: find maximum and minimum on the boundary:

- On the edge from $(-3,0)$ to $(3,0)$, $f = x^2$, hence the maximum is 9 and the minimum is 0.
- On the edge from $(-3,5)$ to $(3,5)$, $f = x^2 + 5x - 5$. f reaches the minimum $-\frac{45}{4}$ when $x = -\frac{5}{2}$ and reaches maximum 19 when $x = 3$.
- On the edge from $(-3,0)$ to $(-3,5)$, $f = y^2 - 9y + 9$. f reaches the minimum $-\frac{45}{4}$ when $y = \frac{9}{2}$ and reaches maximum 9 when $y = 0$.
- On the edge from $(3,0)$ to $(3,5)$, $f = y^2 - 3y + 9$. f reaches the minimum $\frac{27}{4}$ when $y = \frac{3}{2}$ and reaches maximum 9 when $y = 19$.

Putting them together we have the maximum is 19 and minimum is -12. $\square$

Problem 41. It is easier to find the smallest squared distance. $d^2 = (x-2)^2 + y^2 + (z+3)^2$ for $x+y+z = 1$. Since $z = 1 - x - y$, $d^2 = (x-2)^2 + y^2 + (4-x-y)^2$. The critical point of d^2 can be found by solving

$$\begin{cases} 2(x-2) - 2(4-x-y) &= 0 \\ 2y - 2(4-x-y) &= 0 \end{cases}$$

The solution is $(\frac{8}{3}, \frac{2}{3})$. Since for x, y big, f is very big. Hence the critical point $(\frac{8}{3}, \frac{2}{3})$ must correspond to the global minimum. Hence the minimum distance is $\sqrt{\frac{4}{3}}$. $\square$

Problem 44. Let the three numbers be x, y, z, then we want to maximize $f = xyz$ for $x + y + z = 100$ and $x > 0, y > 0, z > 0$. Substitute $z = 100 - x - y$, we have $f = xy(100 - x - y)$ for $x > 0, y > 0$ and $x + y < 100$.

The critical point of f is $(\frac{100}{3}, \frac{100}{3})$, which is inside the domain, and $f(\frac{100}{3}, \frac{100}{3}) = \frac{1000000}{27}$. When f restricted to the boundary of the domain, the value is 0. So the maximum is $\frac{1000000}{27}$. $\square$

Problem 51. Assume the dimensions of the box are x, y, z. We need to maximize xyz given $4(x+y+z) = c$. This is exactly the situation in the previous question(if $c = 400$). Therefore the maximum $\frac{c^3}{12^3}$ is achieved when $x = y = z = \frac{c}{12}$. $\square$

Problem 52. Assume the dimensions of the aquarium is x, y, z. Assume glass costs 1 per unit. Then we need to minimize $2yz + 2xz + 5xy$ given $xyz = V$. Since $z = \frac{V}{xy}$, $f = \frac{2V}{x} + \frac{2V}{y} + 5xy$ for $x, y > 0$.

Since $\nabla f = \langle -\frac{2V}{x^2} + 5y, -\frac{2V}{y^2} + 5x \rangle$, there is only one critical point $(\sqrt[3]{\frac{2V}{5}}, \sqrt[3]{\frac{2V}{5}})$. Note that if $x \to 0$, then $yz \to \infty$, if $y \to 0$, then $xz \to \infty$ and if $z \to 0$, then $xy \to \infty$. That is f diverges to ∞ when any one of x, y, z approaches 0. Therefore the unique critical point must correspond to the global minimum. So the optimal dimensions are $\sqrt[3]{\frac{2V}{5}}, \sqrt[3]{\frac{2V}{5}}, \sqrt[3]{\frac{25V}{4}}$ $\square$

14.8

Problem 1. At any solution to the Larangian multiplier, we must have ∇f is perpendiculer to the constriant $g(x,y) = 8$. In other words, the level curves of f is tangent to $g(x,y) = 8$. From the picture we find four points satifying such condition. The maximum value among is 60 and the minimum value is 30. $\square$

Problem 3. By Lagrange multipliers we have

$$\begin{cases} 2x &= \lambda 2x \\ -2y &= \lambda 2y \\ x^2 + y^2 &= 1 \end{cases}$$

If $x \neq 0$, then from the first equation $\lambda = 1$, hence $-2y = 2y$, i.e. $y = 0$. Therefore we have two solutions $(1, 0)$ and $(-1, 0)$. If $x = 0$, then $y \neq 0$ by the third equation, hence $\lambda = -1$ and $y = \pm 1$, we have the other two solutions $(0, -1), (0, 1)$. Plugging them in, we have the maximum is 1 and the minimum is -1. $\square$

Problem 11. By Lagrange multipliers we have

$$\begin{cases} 2x &= \lambda 4x^3 \\ 2y &= \lambda 4y^3 \\ 2z &= \lambda 4z^3 \\ x^4 + y^4 + z^4 &= 1 \end{cases}$$

If x, y, z are nonzero, the first three equastions imply that $x^2 = y^2 = z^2 = \frac{1}{2\lambda}$. Combining with the third equation, we have eight solutions $(\pm\sqrt[4]{\frac{1}{3}}, \pm\sqrt[4]{\frac{1}{3}}, \pm\sqrt[4]{\frac{1}{3}})$.

If only one of x, y, z is zero, assume it is x, then by the second and third equation, we have $y^2 = z^2$. So there are four solutions $(0, \pm\sqrt[4]{\frac{1}{2}}, \pm\sqrt[4]{\frac{1}{2}})$. If $y = 0$, we have another four $(\pm\sqrt[4]{\frac{1}{2}}, 0, \pm\sqrt[4]{\frac{1}{2}})$ and if $z = 0$, we have $(\pm\sqrt[4]{\frac{1}{2}}, \pm\sqrt[4]{\frac{1}{2}}, 0)$

If two of z, y, z are zero, then the solutions are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$

Plugging those points into the function, we have the maximum is $\sqrt{3}$ and minimum is 1.

$\square$

Problem 16. By Lagrange multipliers we have

$$\begin{cases} 2x &= \lambda \\ 4y &= 2\lambda \\ 6z &= 3\lambda \\ x + 2y + 3z &= 10 \end{cases}$$

The first three euqations imply that $x = y = z$. Hence the solution is $(\frac{10}{3}, \frac{10}{3}, \frac{10}{3})$. If any one of x, y, z is very big, f is also very big. Therefore f has no maximum, and the solution from the Lagrange multipliers corresponds to the global minimum $\frac{200}{3}$. $\square$

Problem 23. First step: find the interior critical points. Since $\nabla f = \langle -ye^{-xy}, -xe^{-xy} \rangle$, the only critical point is $(0, 0)$, which is inside the domain. $f(0, 0) = 1$.

Second step: find the maximum and minimum on the boundary. The boundary is $x^2 + 4y^2 = 1$ and we will use Lagrange multipliers:

$$\begin{cases} -ye^{-xy} &= 2\lambda x \\ -xe^{-xy} &= 8\lambda y \\ x^2 + 4y^2 &= 1 \end{cases}$$

If we multple x to the first equation and y to the second equation, we have $-xye^{-xy} = 2\lambda x^2 = 8\lambda y^2$. If $\lambda \neq 0$, then we know $x^2 = 4y^2$. Hence the solutions are $(\pm\sqrt{\frac{1}{2}}, \pm\frac{1}{2}\sqrt{\frac{1}{2}})$. If the $\lambda = 0$, then the first two equations imply that $x = y = 0$, contradicting the last equation. Evaluating the function on those four points, we have the maximum on the boundry is $e^{\frac{1}{4}}$ and the minimum is $e^{-\frac{1}{4}}$

Putting them together, the global maximum is $e^{\frac{1}{4}}$ and the global minimum is $e^{-\frac{1}{4}}$. $\square$

Problem 29. Assume the dimensions of the rectangle are x, y. We need to maximize $A = xy$ subject to constraints $2x + 2y = p$, $x \geq 0$ and $y \geq 0$. Geometrically, the constraints give a line segement. To find the maximum, we need to use Lagrange multipliers to locate critical points in the interior of the line segement and evaluating the function there, we also need to compare them with the boundary value.

Step one: Lagrange multipliers for the interior critical points:

$$\begin{cases} y &= 2\lambda \\ x &= 2\lambda \\ 2x+2y &= p \end{cases}$$

Hence the critical point is $(\frac{p}{4}, \frac{p}{4})$, where $A = \frac{p^2}{16}$.

Step two: Find the maximum and minimum on the boundary. Since the boundary is two points $(0, \frac{p}{2})$ and $(\frac{p}{2}, 0)$, the boundary maximum and minimum are both 0.

Therefore the global maximunm is $\frac{p^2}{16}$, and it is reached when $x = y = \frac{p}{2}$. □

Problem 30. It is easier to maximize $A^2 = s(s-x)(s-y)(s-z)$ subject to constraints $x+y+z = p$, $x \geq 0, y \geq 0, z \geq 0, x+y \geq z, y+z \geq x$ and $x+z \geq y$. The last three inequalities are necessary conditions for x, y, z to form a triangle. The domain for x, y, z is actually a triangle in space, where the edges of triangle come from the inequalities.

Step one: Lagrange multipliers for the interior critical points. Note that $s = \frac{p}{2}$ is a constant.

$$\begin{cases} -s(s-y)(s-z) &= \lambda \\ -s(s-x)(s-z) &= \lambda \\ -s(s-x)(s-y) &= \lambda \\ x+y+z &= p \end{cases}$$

Multiply $(s-x)$ to the first equation, multiply $(s-y)$ to the second and $(s-z)$ to the third. If $\lambda \neq 0$, we have $s-x = s-y = s-z = -\frac{s(s-x)(s-y)(s-z)}{\lambda}$. Since $x+y+z = 2s = p$, we must have $x = y = z = \frac{p}{3}$. In this case, the area is $\frac{\sqrt{3}p^2}{36}$. If $\lambda = 0$, from the first three equations, we must have two of $(s-x), (s-y), (s-z)$ are zero, no matter what are the solutions, the area must be zero by Heron's formula.

Step two: the boundary maximum. Since the boundary points are thoses points (x, y, z) on the plane $x+y+z = p$ such that one of the six inequalities becomes equality. Once we have a equality, the triangle collapses to a line segement, hence the area must be zero.

Therefore the global maximum comes from the interior critical point $(\frac{p}{3}, \frac{p}{3}, \frac{p}{3})$. □

Problem 31. We will maximize the distance square $d^2 = (x-2)^2 + y^2 + (z+3)^2$ By Lagrange multipliers we have

$$\begin{cases} 2(x-2) &= \lambda \\ 2y &= \lambda \\ 2(z+3) &= \lambda \\ x+y+z &= 1 \end{cases}$$

From the first three equations, we know that $(x-2) = y = (z+3)$. Hence the only solution is $(\frac{8}{3}, \frac{2}{3}, -\frac{7}{3})$, where the distance is $\sqrt{\frac{4}{3}}$. When any one of x, y, z is big, the distance is also big. Therefore, the only critical point we find above corresponds to the global minimum. □

Problem 37. Since we can rotate the sphere as well as the box inside it, it is safe to assume all the edges of the box are parallel to the axes. Assume the coordinate of the vertex in the octant is (x, y, z), we need to maximze $V = 8xyz$ subject to $x^2 + y^2 + z^2 = r^2$, $x \geq 0, y \geq 0$ and $z \geq 0$.

Step one: Use Lagrange multipliers to locate interior critical points.

$$\begin{cases} 8yz &= 2\lambda x \\ 8xz &= 2\lambda y \\ 8xy &= 2\lambda z \\ x^2+y^2+z^2 &= r^2 \end{cases}$$

Multiply the first eqaution by x, the second by y and the third by z. We have $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2 = 8xyz$. If $\lambda \neq 0$, we must have $x^2 = y^2 = z^2$. Since we also have $x, y, z \geq 0$, the only solution is $(r\sqrt{\frac{1}{3}}, r\sqrt{\frac{1}{3}}, r\sqrt{\frac{1}{3}})$, where $V = 8r^3\sqrt{\frac{1}{27}}$. If $\lambda = 0$, then the first three equations imply that two of x, y, z are zero, and the volume is 0 since the box collapses.

Step two: the boundary maximum. Since the boundary of the contraints come from $x = 0$ or $y = 0$ or $z = 0$. In any case, the box collapses hence the volume is 0.

Therefore the maximum is $8r^3\sqrt{\frac{1}{27}}$.

One can avoid the discussion of the boundary by considering maximizing $V^2 = 64x^2y^2z^2$ for any point x, y, z on the sphere $x^2 + y^2 + z^2 = r^2$. The reason of using V^2 instead of V is that $V = 8|xyz|$ is not differentiable unless we restrict the function to the first octant like what we did in the solution. $\square$