Solutions to Homework 4

14.5

1.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) = 2t(y^3 - 2xy + 3xy^2 - x^2)$$

5.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = 2te^{y/z} - xe^{y/z}\frac{1}{z} + 2xe^{y/z}(-\frac{y}{z^2}) = e^{y/z}(2t - \frac{x}{z} - \frac{2xy}{z^2})$$

14.

$$R_s(1,2) = G_u(u(1,2),v(1,2))u_s(1,2) + G_v(u(1,2),v(1,2))v_s(1,2) = 9 \times 4 + (-2) \times 2 = 32$$

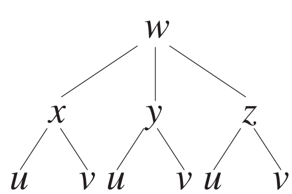
$$R_t(1,2) = G_u(u(1,2),v(1,2))u_t(1,2) + G_v(u(1,2),v(1,2))v_t(1,2) = 9 \times (-3) + (-2) \times 6 = -39$$

16.

$$g_r(1,2) = f_x(0,0)(2) + f_y(0,0)(-4) = 8 - 32 = -24$$

$$g_s(1,2) = f_x(0,0)(-1) + f_y(0,0)(4) = -4 + 32 = 28$$

18.



23.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y+z)\cos\theta + (x+z)\sin\theta + (x+y)\theta = 0 + \pi + \pi = 2\pi$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = -(y+z)r\sin\theta + (x+z)r\cos\theta + (x+y)r = -(2+\pi) \times 2 + 0 + 2 \times 2 = -2\pi$$

27.

Taking x derivative on both sides, we obtain

$$-y\sin\theta + y_x\cos x = 2x + 2yy_x$$

$$y_x = \frac{2x + y\sin\theta}{\cos x - 2y}$$

33.

Taking x derivative on both sides, we obtain

$$e^z z_x = yz + xyz_x$$
$$z_x = \frac{yz}{e^z - xy}$$

Taking y derivative on both sides, we obtain

$$e^z z_y = xz + xyz_y$$
$$z_y = \frac{xz}{e^z - xy}$$

35.

We first know that $t = 3$ for $x = 2$ and $y = 3$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = T_x(x,y)\frac{1}{2\sqrt{t+1}} + T_y(x,y)(\frac{1}{3}) = 4 \times \frac{1}{4} + 3 \times \frac{1}{3} = 2^{\circ}\text{C/s}$$

39.

(a)
The volume is $V = lwh$

$$\frac{dV}{dt} = \frac{\partial V}{\partial l}\frac{\partial l}{\partial t} + \frac{\partial V}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial V}{\partial h}\frac{\partial h}{\partial t} = 2 \times 2 \times 2 + 1 \times 2 \times 2 + 1 \times 2 \times (-3) = 6\text{m}^3\text{/s}$$

(b)
The surface area is $A = 2lw + 2lh + 2wh$

$$\frac{dA}{dt} = \frac{\partial A}{\partial l}\frac{\partial l}{\partial t} + \frac{\partial A}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial A}{\partial h}\frac{\partial h}{\partial t} = 2 \times (2+2) \times 2 + 2 \times (1+2) \times 2 + 2 \times (1+2) \times (-3) = 10\text{m}^2\text{/s}$$

(c)
The length of a diagonal is $d = \sqrt{l^2 + w^2 + h^2}$

$$\frac{dd}{dt} = \frac{\partial d}{\partial l}\frac{\partial l}{\partial t} + \frac{\partial d}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial d}{\partial h}\frac{\partial h}{\partial t} = \frac{1}{\sqrt{1+4+4}} \times 2 + \frac{2}{\sqrt{1+4+4}} \times 2 + \frac{2}{\sqrt{1+4+4}} \times (-3) = 0\text{m/s}$$

45.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = f_x \cos\theta + f_y \sin\theta$$
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -f_x r \sin\theta + f_y r \cos\theta$$

It can be shown that

$$(\frac{\partial z}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2 = (f_x \cos\theta + f_y \sin\theta)^2 + \frac{1}{r^2}(-f_x r \sin\theta + f_y r \cos\theta)^2 = f_x^2 + f_y^2 = (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2$$

52.

Please refer to P45 for part (a) and (b).
(c)
Starting from

$$\frac{\partial z}{\partial r} = f_x \cos\theta + f_y \sin\theta$$

we take the θ derivative of $\frac{\partial z}{\partial r}$ and get

$$\begin{aligned}\frac{\partial^2 z}{\partial r \partial \theta} &= -f_x \sin\theta + f_y \cos\theta + (-f_{xx} r \sin\theta + f_{xy} r \cos\theta)\cos\theta + (-f_{yx} r \sin\theta + f_{yy} r \cos\theta) sin\theta \\ &= -f_x \sin\theta + f_y \cos\theta + r(-f_{xx} \sin\theta\cos\theta + f_{xy}\cos^2\theta - f_{xy}\sin^2\theta + f_{yy}\sin\theta\cos\theta)\end{aligned}$$

Note that $\frac{\partial z}{\partial r}$ is explicit in θ, that's why the first two terms have the first order derivatives.

53.

$$\frac{\partial^2 z}{\partial r^2} = f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta$$

Starting from

$$\frac{\partial z}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta$$

we take the θ derivative of $\frac{\partial z}{\partial \theta}$ and get

$$\frac{\partial^2 z}{\partial \theta^2} = r^2(f_{xx}\sin^2\theta - 2f_{xy}\sin\theta\cos\theta + f_{yy}\cos^2\theta) - r(f_x\cos\theta + f_y\sin\theta)$$

$$LHS = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

$$= f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta + \frac{1}{r^2}[r^2(f_{xx}\sin^2\theta - 2f_{xy}\sin\theta\cos\theta + f_{yy}\cos^2\theta) - r(f_x\cos\theta + f_y\sin\theta)] + \frac{1}{r}(f_x\cos\theta + f_y\sin\theta)$$

$$= f_{xx} + f_{yy} = RHS$$

14.6

7.

(a)

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j}$$

(b)

$$\nabla f(2,1) = \mathbf{i} - 2\mathbf{j}$$

(c)

$$D_{\mathbf{u}} f(2,1) = (\mathbf{i} - 2\mathbf{j}) \cdot (\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}) = -1$$

9.

(a)

$$\nabla f(x,y,z) = < 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2 >$$

(b)

$$\nabla f(2,-1,1) = < -3, 2, 2 >$$

(c)

$$D_{\mathbf{u}} f(2,-1,1) = < -3,2,2 > \cdot < 0, \frac{4}{5}, -\frac{3}{5} > = \frac{2}{5}$$

24.

$$\nabla f(x,y,z) = < \ln(yz), \frac{x}{y}, \frac{x}{z} > = < 0, \frac{1}{2}, 2 >$$

The maximum rate of change is $|\nabla f(x,y,z)| = \sqrt{\frac{17}{4}}$, in the direction of the gradient $< 0, \frac{1}{2}, 2 >$.

27.

(a)
Given $D_{\mathbf{u}} f = |\nabla f| \cos\theta$, its minimum occurs at $\theta = \pi$, i.e., $D_{\mathbf{u}} f = -|\nabla f|$. The corresponding direction is $-\nabla f$.
(b)

$$\nabla f(x,y) =< 4x^3y - 2xy^3, x^4 - 3x^2y^2 >$$
$$-\nabla f(2,-3) =< -12, 92 >$$

34.

(a)

$$\nabla f(x,y) =< -0.01x, -0.02y >=< -0.6, -0.8 >$$

The direction due south is $\mathbf{u} =< 0, -1 >$ and the directional derivative is

$$D_{\mathbf{u}} f(x,y) =< -0.6, -0.8 > \cdot < 0, -1 >= 0.8$$

which means you start to ascend at the rate 0.8m/s.
(b)
The direction northwest is $\mathbf{u} =< -1, 1 >$ and the directional derivative is

$$D_{\mathbf{u}} f(x,y) =< -0.6, -0.8 > \cdot \frac{< -1, 1 >}{\sqrt{2}} = -0.14$$

which means you start to descend at the rate 0.14m/s.
(c)

The slope is the largest in the direction of the gradient $< -0.6, -0.8 >$, its angle to the horizontal axis is $\cos^{-1}(0.6)$.

$$|\nabla f(x,y)| = 1$$

40.

(a)

$$D_{\mathbf{u}} f =< f_x, f_y > \cdot < a, b >= af_x + bf_y$$

$$D_{\mathbf{u}}^2 f =< \frac{\partial D_{\mathbf{u}} f}{\partial x}, \frac{\partial D_{\mathbf{u}} f}{\partial y} > \cdot < a, b >=< af_xx + af_{xy}, bf_{xy} + bf_{yy} > \cdot < a, b >= a^2 f_{xx} + 2ab f_{xy} + b^2 f_{yy}$$

(b)

$$f_x = e^{2y}$$
$$f_y = 2xe^{2y}$$
$$f_{xx} = 0$$
$$f_{yy} = 4xe^{2y}$$
$$f_{xy} = 2e^{2y}$$

Based on the formula in (a)

$$D_{\mathbf{u}}^2 f = 16 \times 0 + 2 \times 24(2e^{2y}) + 36(4xe^{2y}) = (96 + 144x)e^{2y}$$

42.

$$F(x,y,z) = -x + y^2 + z^2 + 1$$

The normal direction is $< F_x, F_y, F_z >=< -1, 2y, 2z >=< -1, 2, -2 >$, therefore the tangent plane is

$$-(x-3) + 2(y-1) - 2(z+1) = 0$$

and the normal line is

$$\frac{x-3}{-1} = \frac{y-1}{2} = \frac{z+1}{-2}$$

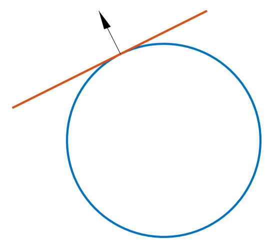
50.

The gradient vector

$$\nabla g(1,2) = < 2x-4, 2y > = < -2, 4 >$$

The tangent line is

$$-2(x-1) + 4(y-2) = 0$$



55.

The normal direction of the hyperboloid is $< 2x, -2y, -2z >$. The vector should be parallel to $< 1, 1, -1 >$. So the points should be, if exist,

$$\frac{2x}{1} = \frac{-2y}{1} = \frac{-2z}{-1}$$

Substituting of the parametric form of the line equation $y = -x,\ z = x$ into the equation of the hyperboloid, we obtain $-x^2 = 1$ which is impossible. So there are no such points.

56.

The normal direction of the ellipsoid is $< 6x, 4y, 2z > = < 6, 4, 4 >$ and that for the sphere is $< 2x-8, 2y-6, 2z-8 > = < -6, -4, -4 >$, so they have the same tangent plane at the point $(1, 1, 2)$.

63.

The normal direction of the paraboloid at the point $(-1, 1, 2)$ is $< 2x, 2y, -1 > = < -2, 2, -1 >$ and that of the ellipsoid is $< 8x, 2y, 2z > = < -8, 2, 4 >$. The tangent vector of the intersection curve should be perpendicular to both of the normal directions. So it can be obtained by taking the cross product.

$$< -2, 2, -1 > \times < -8, 2, 4 > = < 10, 16, 12 >$$

The tangent line can then be written as

$$\frac{x+1}{10} = \frac{x-1}{16} = \frac{x-2}{12}$$

65.

Substitution of the parametric equation of the helix into the paraboloid equation obtains $t = 1$. The point they intersect is $(-1, 0, 1)$. The tangent vector of the helix at this point is $< -\pi \sin \pi t, \pi \cos \pi t, 1 > = < 0, -\pi, 1 >$. The normal direction of the paraboloid at this point is $< 2x, 2y, -1 > = < -2, 0, -1 >$. Then the angle between the two vectors is

$$\cos\theta = \frac{< 0, -\pi, 1 > \cdot < -2, 0, -1 >}{| < 0, -\pi, 1 > || < -2, 0, -1 > |} = \frac{-1}{\sqrt{5\pi^2 + 1}}$$

Therefore,

$$\theta = \cos^{-1} \frac{-1}{\sqrt{5\pi^2 + 1}} \approx 97.8^\circ$$

So the angle between the tangent plane and the helix is $\theta - \pi/2 = 7.8^\circ$.