Solutions to Homework 4

14.5

1. 
\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) = 2t(y^3 - 2xy + 3xy^2 - x^2)
\]

5. 
\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = 2te^{y/z} - xe^{y/z} \frac{1}{z} + 2xe^{y/z} \left( - \frac{y}{z^2} \right) = e^{y/z} \left( 2t - \frac{x}{z} - \frac{2xy}{z^2} \right)
\]

14. 
\[
R_u(1,2) = G_u(u(1,2), v(1,2))u_u(1,2) + G_v(u(1,2), v(1,2))v_u(1,2) = 9 \times 4 + (-2) \times 2 = 32
\]
\[
R_v(1,2) = G_u(u(1,2), v(1,2))u_v(1,2) + G_v(u(1,2), v(1,2))v_v(1,2) = 9 \times (-3) + (-2) \times 6 = -39
\]

16. 
\[
g_r(1,2) = f_x(0,0)(2) + f_y(0,0)(-4) = 8 - 32 = -24
\]
\[
g_s(1,2) = f_x(0,0)(-1) + f_y(0,0)(4) = -4 + 32 = 28
\]

18. 

23. 
\[
\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} = (y+z) \cos \theta + (x+z) \sin \theta + (x+y) \theta = 0 + \pi + \pi = 2\pi
\]
\[
\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \theta} = -(y+z) r \sin \theta + (x+z) r \cos \theta + (x+y) r = -(2+\pi) \times 2 + 0 + 2 \times 2 = -2\pi
\]

27. 
Taking x derivative on both sides, we obtain
\[
-ysin\theta + y_x \cos x = 2x + 2yy_x
\]
\[
y_x = \frac{2x + y \sin \theta}{\cos x - 2y}
\]
33.

Taking $x$ derivative on both sides, we obtain
\[ e^{z}z_x = yz + xz_x \]
\[ z_x = \frac{yz}{e^{z} - xy} \]

Taking $y$ derivative on both sides, we obtain
\[ e^{z}z_y = xz + xz_y \]
\[ z_y = \frac{xz}{e^{z} - xy} \]

35.

We first know that $t = 3$ for $x = 2$ and $y = 3$

\[ \frac{dT}{dt} = \frac{dT}{dx} \frac{dx}{dt} + \frac{dT}{dy} \frac{dy}{dt} = T_0(x,y) \left( \frac{1}{2\sqrt{1 + 1}} + T_0(x,y)(\frac{1}{3}) \right) = 4 \times \frac{1}{4} + 3 \times \frac{1}{3} = 2^\circ C/s \]

39.

(a)
The volume is $V = lwh$

\[ \frac{dV}{dt} = \frac{\partial V}{\partial l} \frac{dl}{dt} + \frac{\partial V}{\partial w} \frac{dw}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = 2 \times 2 \times 2 + 1 \times 2 \times 3 \times 2 \times (-1) = 6 m^3/s \]

(b)
The surface area is $A = 2lw + 2lh + 2wh$

\[ \frac{dA}{dt} = \frac{\partial A}{\partial l} \frac{dl}{dt} + \frac{\partial A}{\partial w} \frac{dw}{dt} + \frac{\partial A}{\partial h} \frac{dh}{dt} = 2 \times (2 + 2) \times 2 + 2 \times (1 + 2) \times 2 \times (1 + 2) \times (-3) = 10 m^2/s \]

(c)
The length of a diagonal is $d = \sqrt{l^2 + w^2 + h^2}$

\[ \frac{dd}{dt} = \frac{\partial d}{\partial l} \frac{dl}{dt} + \frac{\partial d}{\partial w} \frac{dw}{dt} + \frac{\partial d}{\partial h} \frac{dh}{dt} = \frac{1}{\sqrt{1^2 + 4^2}} \times 2 + \frac{2}{\sqrt{1^2 + 4^2}} \times 2 + \frac{2}{\sqrt{1^2 + 4^2}} \times (-3) = 0 m/s \]

45.

\[ \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = f_x \cos \theta + f_y \sin \theta \]

\[ \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = -f_x r \sin \theta + f_y r \cos \theta \]

It can be shown that

\[ \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2 = (f_x \cos \theta + f_y \sin \theta)^2 + \frac{1}{r^2} (-f_x r \sin \theta + f_y r \cos \theta)^2 = f_x^2 + f_y^2 = \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \]

52.

Please refer to P45 for part (a) and (b).

(c)

Starting from

\[ \frac{\partial z}{\partial r} = f_x \cos \theta + f_y \sin \theta \]

we take the $\theta$ derivative of $\frac{\partial z}{\partial r}$ and get

\[ \frac{\partial^2 z}{\partial r \partial \theta} = -f_x \sin \theta + f_y \cos \theta + (f_y r \cos \theta - f_x r \sin \theta) \cos \theta + (f_y r \cos \theta + f_x r \sin \theta \sin \theta) \sin \theta \]

Note that $\frac{\partial z}{\partial r}$ is explicit in $\theta$, that's why the first two terms have the first order derivatives.
\[ \frac{\partial^2 z}{\partial r^2} = f_{xx} \cos^2 \theta + 2f_{xy} \sin \theta \cos \theta + f_{yy} \sin^2 \theta \]

Starting from
\[ \frac{\partial z}{\partial \theta} = -f_x r \sin \theta + f_y r \cos \theta \]
we take the \( \theta \) derivative of \( \frac{\partial z}{\partial \theta} \) and get
\[ \frac{\partial^2 z}{\partial \theta^2} = r^2 (f_{xx} \sin^2 \theta - 2f_{xy} \sin \theta \cos \theta + f_{yy} \cos^2 \theta) - r(f_x \cos \theta + f_y \sin \theta) \]

\[ \text{LHS} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r} = f_{xx} \cos^2 \theta + 2f_{xy} \sin \theta \cos \theta + f_{yy} \sin^2 \theta + \frac{1}{r^2} [r^2 (f_{xx} \sin^2 \theta - 2f_{xy} \sin \theta \cos \theta + f_{yy} \cos^2 \theta) - r(f_x \cos \theta + f_y \sin \theta)] + \frac{1}{r} (f_x \cos \theta + f_y \sin \theta) = f_{xx} + f_{yy} = \text{RHS} \]

14.6
7.
(a)
\[ \nabla f(x,y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = \frac{1}{y} \mathbf{i} - \frac{x}{y^2} \mathbf{j} \]
(b)
\[ \nabla f(2,1) = \mathbf{i} - 2\mathbf{j} \]
(c)
\[ D_u f(2,1) = (\mathbf{i} - 2\mathbf{j}) \cdot (\frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}) = -1 \]
9.
(a)
\[ \nabla f(x,y,z) = < 2xyz - yz^3, x^2z - x^3, x^2y - 3xyz^2 > \]
(b)
\[ \nabla f(2,1,1) = < -3,2,2 > \]
(c)
\[ D_u f(2,1,1) = < -3,2,2 > \cdot < 0, \frac{4}{5}, -\frac{3}{5} > = \frac{2}{5} \]
24.
\[ \nabla f(x,y,z) = < \ln(yz), y, z > = < 0, \frac{1}{2}, 2 > \]
The maximum rate of change is \( |\nabla f(x,y,z)| = \sqrt{\frac{17}{4}} \), in the direction of the gradient \(< 0, \frac{1}{2}, 2 > \).
27.  
(a) Given $D_u f = |\nabla f| \cos \theta$, its minimum occurs at $\theta = \pi$, i.e., $D_u f = -|\nabla f|$. The corresponding direction is $-\nabla f$.

(b) 
$$\nabla f(x,y) = <4x^3y - 2xy^3, x^4 - 3x^2y^2>$$  
$$-\nabla f(2,-3) = <-12,92>$$

34.  
(a) 
$$\nabla f(x,y) = <-0.01x, -0.02y> = <-0.6, -0.8>$$

The direction due south is $u = <-0, -1>$ and the directional derivative is 
$$D_u f(x,y) = <-0.6, -0.8> \cdot <-0, -1> = 0.8$$

which means you start to ascend at the rate 0.8m/s.

(b) 
The direction northwest is $u = <-1, -1>$ and the directional derivative is 
$$D_u f(x,y) = <-0.6, -0.8> \cdot <-1, -1> = -0.14$$

which means you start to descend at the rate 0.14m/s.

(c) 
The slope is the largest in the direction of the gradient $<-0.6, -0.8>$, its angle to the horizontal axis is $\cos^{-1}(0.6)$.

$$|\nabla f(x,y)| = 1$$

40.  
(a) 
$$D_u f = < f_x, f_y > \cdot < a, b > = a f_x + b f_y$$

$$D_u^2 f = < \frac{\partial D_u f}{\partial x}, \frac{\partial D_u f}{\partial y} > \cdot < a, b > = a^2 f_{xx} + 2ab f_{xy} + b^2 f_{yy}$$

(b) 
$$f_x = e^{2y}$$
$$f_y = 2xe^{2y}$$
$$f_{xx} = 0$$
$$f_{yy} = 4xe^{2y}$$
$$f_{xy} = 2e^{2y}$$

Based on the formula in (a) 
$$D_u^2 f = 16 \times 0 + 2 \times 24(2e^{2y}) + 36(4xe^{2y}) = (96 + 144x)e^{2y}$$

42.  
$$F(x,y,z) = -x + y^2 + z^2 + 1$$

The normal direction is $<F_x, F_y, F_z> = <-1, 2y, 2z> = <-1, 2, -2>$, therefore the tangent plane is 
$$-(x-3) + 2(y-1) - 2(z+1) = 0$$

and the normal line is 
$$\frac{x-3}{-1} = \frac{y-1}{2} = \frac{z+1}{-2}$$
50. The gradient vector \( \nabla g(1, 2) = < 2x - 4, 2y > = < -2, 4 > \)

The tangent line is
\[
-2(x - 1) + 4(y - 2) = 0
\]

55. The normal direction of the hyperboloid is \( < 2x, -2y, -2z > \). The vector should be parallel to \( < 1, 1, -1 > \). So the points should be, if exist,
\[
\frac{2x}{1} = \frac{-2y}{1} = \frac{-2z}{-1}
\]

Substituting of the parametric form of the line equation \( y = -x, z = x \) into the equation of the hyperboloid, we obtain \(-x^2 = 1\) which is impossible. So there are no such points.

56. The normal direction of the ellipsoid is \( < 6x, 4y, 2z > = < 6, 4, 4 > \) and that for the sphere is \( < 2x - 8, 2y - 6, 2z - 8 > = < -6, -4, -4 > \), so they have the same tangent plane at the point \( (1, 1, 2) \).

63. The normal direction of the paraboloid at the point \( (-1, 1, 2) \) is \( < 2x, 2y, -1 > = < -2, 2, -1 > \) and that of the ellipsoid is \( < 8x, 2y, 2z > = < -8, 2, 4 > \). The tangent vector of the intersection curve should be perpendicular to both of the normal directions. So it can be obtained by taking the cross product.
\[
< -2, 2, -1 > \times < -8, 2, 4 > = < 10, 16, 12 >
\]

The tangent line can then be written as
\[
\frac{x + 1}{10} = \frac{x - 1}{16} = \frac{x - 2}{12}
\]

65. Substitution of the parametric equation of the helix into the paraboloid equation obtains \( t = 1 \). The point they intersect is \( (-1, 0, 1) \). The tangent vector of the helix at this point is \( < -\pi \sin \pi t, \pi \cos \pi t, 1 > = < 0, -\pi, 1 > \). The normal direction of the paraboloid at this point is \( < 2x, 2y, -1 > = < -2, 0, -1 > \). Then the angle between the two vectors is
\[
\cos \theta = \frac{< 0, -\pi, 1 > \cdot < -2, 0, -1 >}{|| < 0, -\pi, 1 > || \cdot || < -2, 0, -1 > ||} = \frac{-1}{\sqrt{5\pi^2 + 1}}
\]

Therefore,
\[
\theta = \cos^{-1} \left( \frac{-1}{\sqrt{5\pi^2 + 1}} \right) \approx 97.8^\circ
\]

So the angle between the tangent plane and the helix is \( \theta - \frac{\pi}{2} = 7.8^\circ \).