Homework solutions

math 53

February 7, 2018

Section 14.1:
14. We are asked to find and sketch the domain of the function

$$\sqrt[4]{x-3y}$$

This function is defined whenever $x - 3y \geq 0$
This is equivalent to the condition

$$\vec{v} \cdot < 1, -3 > \geq 0$$

which describes the half-plane of vectors in $\mathbb{R}^2$ forming an angle less than or equal to $\pi/2$ with the vector

$$< 1, -3 >$$

(see attached drawing)

30. (see attached drawing)

32. a) III

$$f(x,y) = \frac{1}{1+x^2+y^2}$$

It is clear that $f \leq 1$ everywhere, attains this maximum only at the origin where $x^2 + y^2 = 0$, and since $x^2 + y^2$ is the square of the distance from the origin, f is constant on circles centered at the origin. only III satisfies these conditions.

b) I

$$f(x,y) = \frac{1}{1+x^2y^2}$$

Again, $f \leq 1$ everywhere and equals one only where $x^2y^2 = 0$, which occurs along the x and y axes and nowhere else. Therefore f attains its maximum everywhere along the x and y axes and nowhere else. Only I and II satisfy this condition. If we look only at points where $y = x$, the equation for f simplifies to

$$\frac{1}{1+x^4}$$

which strictly decreases as we increase $|x|$. This means if we move away from the origin along the line $y = x$, the graph should never begin to increase, which rules out II, where we see slight increasing occurring at the edge of the region shown.

c) IV

$$f(x,y) = ln(x^2 + y^2)$$

this is the only function in the collection that becomes unbounded near $(0,0)$. IV is similarly the only graph that diverges near the origin.

d) V

$$f(x,y) = cos(\sqrt{x^2 + y^2})$$

This function is constant along circles centered at the origin and oscillates as we move away from the origin. Only V satisfies this condition.

e) VI

$$f(x,y) = |xy|$$

This function is zero only along the x and y axes and greater than zero everywhere else. Only VI satisfies this.

f) II

$$f(x,y) = cos(xy)$$

On the one hand of course II is the only choice left, but its also clear that II is correct because $f = 1$ where $xy = 0$ and, similarly to b), if we look where $y = x$, the equation reduces to

$$cos(x^2)$$

and so we expect to see oscillation as we move away from the origin along the line $y = x$, and we can see this occurring in II.

36. The first image, where the rings occur closer and closer together as we move from the origin, corresponds to the paraboloid. One way of seeing this is to take, for instance, the paraboloid

$$f(x,y) = x^2 + y^2$$

and consider the set of points in the plane where $f(x,y) = c$ for some fixed choice of $c > 0$. This set is the same as the set of points satisfying

$$\sqrt{x^2 + y^2} = \sqrt{c}$$

i.e. a circle of radius $\sqrt{c}$.
Now if we look at a collection of such sets, say first the set where $f(x,y) = 1$, then the set where $f(x,y) = 2$, then $f(x,y) = 3$ and so on, we will see concentric circles first of radius 1, then radius $\sqrt{2}$, then $\sqrt{3}$ and so on. The square root function $\sqrt{x}$ grows more and more slowly as $x > 0$ increases. This corresponds to the characteristic that the circles appear closer and closer together as we move farther from the origin.

38. see attached drawing

46. see attached drawing

54. see attached drawing

61. C, II

$$z = sin(xy)$$

this function is constant where xy is constant, i.e. where

$$xy = c$$

or

$$y = c/x$$

only C and II exhibit this behavior.

62. A, IV

$$z = e^x cos(y)$$

If we only vary x, z behaves like the exponential function, in particular it will increase without bound for very positive x and tend to zero for very negative x, and if we only vary y, z oscillates. Only A exhibits this behavior.
When $y = (2n+1)\pi/2$, $cos(y) = 0$ and so $z = 0$. Therefore no contour line can cross the lines $y = (2n+1)\pi/2$, which are an infinite collection of evenly space lines parallel to the x axis. Only image IV could satisfy this condition.

63. F, I

$$z = sin(x-y)$$

z is constant along lines $y = x + c$. Only F and I allow for this.

64. E, III

$$z = sin(x) - sin(y)$$

z oscillates when we vary only x and when we vary only y. Only E exhibits this behavior. Looking at E, there are evenly spaced local maxima and minima, little hills and valleys, and each hill for instance will correspond to a collection of concentric loops in the contour map like we see only in III.

65. B, VI

$$z = (1-x^2)(1-y^2)$$

z=1 at the origin and z=0 anywhere on the square determined by the equations $y = 1$, $y = -1$, $x = 1$ nd $x = -1$. This is enough to see that only B and VI could be correct choices.

66. D, V

$$z = \frac{x-y}{1+x^2+y^2}$$

for large values of x and y the denominator in the expression above dominates the numerator and z will be very small. Furthermore z is negative when x=0 and positive when y=0. This is enough to see that only D could be the graph of z. Then we can see that V has two obvious families of concentric closed loops that correspond to the maximum and minimum points visible in graph D.

71. a) The whole graph of f is lifted 2 units to obtain the graph of g.

b) The distance of the graph of f from the x-y plane is doubled at each point, resulting in the stretching of f by a factor of 2, to obtain the graph of g.

c) The graph of f is flipped upside down to obtain the graph of g.

d) The graph of f is flipped upside down and lifted 2 units to obtain the graph of g.

72. a) The graph of f is shifted 2 units in the positive x direction to obtain the graph of g.

b) The graph of f is shifted 2 units in the negative y direction to obtain the graph of g.

c) The graph of f is shifted 3 units in the negative x direction and 4 units in the positive y direction to obtain the graph of g.

Section 14.2:

7. sin is defined on the range of $(x-y)$, so $sin(x-y)$ is continuous. $ysin(x-y)$ is thus a product of continuous functions and is therefore continuous. The limit at $(\pi, \pi/2)$ can therefore be evaluated directly and is $\frac{\pi}{2} sin(\pi/2) = \frac{\pi}{2}$.

9. If we look at the function

$$f(x,y) = \frac{x^4 - 4y^2}{x^2 + 2y^2}$$

along the y axis (where $x = 0$), then $f(x,y) = -2$ everywhere. Along the x axis however, $f(x,y) = x^2$, which approaches zero near the origin. This shows that the limit does not exist.

13.

$$
\begin{aligned}
0 &\le |\frac{xy}{\sqrt{x^2+y^2}}| \\
&\le |\frac{x^2+y^2}{2\sqrt{x^2+y^2}}| \\
&\le \frac{1}{2}\sqrt{x^2+y^2}
\end{aligned}
$$

the function in the last line clearly approaches zero near the origin and gives an upper bound for the magnitude of

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

which shows that f also approaches zero near the origin.

28.

$$f(x,y) = \frac{1}{1-x^2-y^2}$$

f is discontinuous where the denominator above vanishes. If we re-express f as

$$f(x,y) = \frac{1}{1-(\sqrt{x^2+y^2})^2}$$

then, since $\sqrt{x^2+y^2}$ measures distance from the origin, it is clear that the denominator vanishes only on the circle of radius 1 centered at the origin

$$\{(x.y) \in \mathbb{R}^2 | \quad x^2+y^2 = 1)\}$$

Furthermore, if $\sqrt{x^2+y^2} < 1$, the denominator is positive and if $\sqrt{x^2+y^2} > 1$, it is negative, so as we approach the set above from outside, f approaches $-\infty$ but as we approach the circle from the inside, f approaches $+\infty$.

33.

$$G(x,y) = \sqrt{x} + \sqrt{1-x^2-y^2}$$

This function is only defined on the intersection of the unit disk $x^2+y^2 \le 1$ with the half plane $x \ge 0$

$$\{(x,y) \in \mathbb{R}^2 | \quad x^2+y^2 \le 1, \quad x \ge 0\}$$

On this set, both the continuous function x and the continuous function $1-x^2-y^2$ take values in the nonnegative numbers, where $\sqrt{\cdot}$ is defined and continuous. Therefore the function is continuous where it is defined.

45.

$$
\begin{aligned}
|x-a|^2 &= (x-a)\cdot(x-a) \\
&= |x|^2 + |a|^2 - 2(x\cdot a)
\end{aligned}
$$

so

$$2(x \cdot a) = |x|^2 + |a|^2 - |x-a|^2$$

which implies, along with the Cauchy-Schwartz inequality,

$$\begin{aligned}(|x| - |a|)^2 &= |x|^2 + |a|^2 - 2|x||a| \\ &\leq |x|^2 + |a|^2 - 2(x \cdot a) \\ &= |x-a|^2\end{aligned}$$

which shows continuity by definition 5 in 14.2.

46. let $c = < c_1, ..., c_n >$ and $x = < x_1, ..., x_n >$. Then

$$c \cdot x = c_1 x_1 + ... + c_n x_n$$

which is a linear combination of linear functions and is therefore continuous.

Section 14.3:

8. The slope in the x direction is positive at $(1,2)$ and grows steeper as y increases, so the sign of f_{xy} is positive at this point.

18.

$$\begin{aligned}f(x,y) &= \sqrt{3x+4t} \\ f_x(x,y) &= \frac{3}{2\sqrt{3x+4t}} \\ f_y(x,y) &= \frac{2}{\sqrt{4x+4t}}\end{aligned}$$

19.

$$\begin{aligned}z &= ln(x+t^2) \\ z_x &= \frac{1}{x+t^2} \\ z_t &= \frac{2t}{x+t^2}\end{aligned}$$

41.

$$\begin{aligned}R(s,t) &= te^{s/t} \\ R_t(s,t) &= e^{s/t} - \frac{s}{t}e^{s/t} \\ R_t(0,1) &= 1\end{aligned}$$

50.

$$yz + xln(y) = z^2$$

firstly, for $\frac{\partial z}{\partial x}$

$$y\frac{\partial z}{\partial x} + ln(y) = 2z\frac{\partial z}{\partial x}$$

so

$$\frac{\partial z}{\partial x} = \frac{-ln(y)}{y-2z}$$

Then for $\frac{\partial z}{\partial y}$

$$z + y\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$$

so

$$\frac{\partial z}{\partial y} = \frac{-zy-x}{y(y-2z)}$$

56.

$$T = e^{-2r}cos(\theta)$$
$$T_r = -2e^{-2r}cos(\theta)$$
$$T_\theta = -e^{-2r}sin(\theta)$$
$$T_{r\theta} = T_{\theta r} = 2e^{-2r}sin(\theta)$$
$$T_{rr} = 4e^{-2r}cos(\theta)$$
$$T_{\theta\theta} = -e^{-2r}cos(\theta)$$

71.

$$f(x,y,z) = xy^2z^3 + arcsin(x\sqrt{z})$$
$$f_y = 2xyz^3$$
$$f_{yx} = 2yz^3$$
$$f_{yxz} = 6yz^2 = f_{xzy}$$

74. a) Negative. The function decreases as we increase x at p.

d) Negative. As y increases at p, we move to a region where the contour lines are more closely spaced, which corresponds to more rapid change. Since in the x direction the lines decrease in value, this means we would observe a more rapid descent, which corresponds to a negative value of f_{xy}.

101. Clairaut's Theorem says

$$f_{xy} = f_{yx}$$

which immediately gives

$$f_{xyy} = f_{yxy}$$

but then using Clairaut's theorem on the function f_y, we get

$$f_{yxy} = f_{yyx}$$

which completes what we are asked to prove.

Section 14.4:

3.

$$z = e^{x-y}, \quad (2,2,1)$$
$$v_x(2,2,1) =< 1,0,e^{2-2} >=< 1,0,1 >$$
$$v_y(2,2,1) =< 0,1,-e^{2-2} >=< 0,1,-1 >$$
$$v_x \times v_y =< -1,1,1 >$$

so the plane takes the form

$$-x+y+z-d=0$$

plugging in $(x,y,z) = (2,2,1)$ to solve for d, we get the equation

$$-x+y+z-1=0$$

13. $f(x,y) = x^2e^y$ is the product of differentiable functions and is therefore differentiable.

$$f_x = 2xe^y$$
$$f_y = x^2e^y$$

The linearization at $(1,0)$ is

$$L(x,y) = f(1,0) + f_x(x-1) + f_y(y) = 1+2(x-1)+y$$

19. Using the information provided, the linearization is

$$L(x,y) = 6+(x-2)-(y-5)$$

then we can approximate $f(2.2,4.9)$ by

$$L(2.2,4.9) = 6.3$$

31.

$$z = 5x^2+y^2$$
$$\frac{\partial z}{\partial x} = 10x$$
$$\frac{\partial z}{\partial y} = 2y$$

$$dz = 10xdx + 2ydy$$

now we can compute

$$\Delta z = 5(1.05)^2 + 2.1^2 - (5+4) = 5.5125 + 4.41 - 9 = 0.9225$$

$$dz = 10(0.05) + 4(0.1) = 0.9$$

then

$$\Delta z - dz = 0.0225$$

42. $r_1(0) = 2,1,3$ and $r_2(1) = (2,1,3)$. Therefore the vectors $r_1'(0)$ and $r_2'(1)$ are parallel to the plane.

$$r_1'(0) = < 3, 0, -4 >$$

$$r_2'(1) = < 2, 6, 2 >$$

Therefore

$$r_1'(0) \times r_2'(1) = < 24, -14, 18 >$$

is normal to the plane in question. We can scale this vector to $< 12, -7, 9 >$ and it will of course still be normal to the desired plane. Therefore, plugging in the point $p = (2,1,3)$, we see that the plane is described by

$$12x - 7y + 9z - 44 = 0$$

or

$$y = 1$$

46. a) Problem 45 shows that if f is differentiable at a point, then its limit exists at that point. However, if we look at the f along the line $y = x$,

$$f = 1/2$$

but along the line $y = 0$ for instance,

$$f = 0$$

this shows that the function cannot have a well defined limit at the origin, where these two lines intersect. Thus f is not differentiable. However, for any choice of y

$$f(0,y) = 0$$

and for any choice of x

$$f(x,0) = 0$$

These statements respectively imply

$$f_y(0,0) = 0$$

and

$$f_x(0,0) = 0$$

b) If we evaluate f where $y = x$, we get

$$f(x,y) = \frac{x^2}{2x^2} = 1/2$$

but if we evaluate f where $y = 0$ for instance, we get

$$f(x,y) = 0$$

since the two lines $y = x$ and $y = 0$ intersect at the origin, this shows that f cannot be continuous at the origin.