Math 53 Homework 13 Solutions

3. By Stokes' theorem, we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r},$$

where C is the boundary circle $x^2+z^2 = 16, y = 0$, oriented appropriately. We can parametrize this curve via $\mathbf{r}(t) = \langle 4\cos t, 0, -4\sin t\rangle$, so

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \langle -4\sin t, 4\cos t, 0\rangle \cdot \langle -4\sin t, 0, -4\cos t\rangle\; dt \\ &= \int_0^{2\pi} 16\sin^2 t\; dt \\ &= \boxed{16\pi}.\end{aligned}$$

5. Let R be the square with vertices $(\pm 1, \pm 1, -1)$ oriented upward, so that $S \cup R$ is the cube with vertices $(\pm 1, \pm 1, \pm 1)$. Then by Stokes' theorem,

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_R \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$

since they share the same boundary. Now

$$\text{curl}\,\mathbf{F} = \langle x^2 z, xy - 2xyz, y - xz\rangle,$$

and the unit normal to R is just $\langle 0, 0, 1\rangle$, so we have

$$\begin{aligned}\iint_R \text{curl}\,\mathbf{F}\cdot d\mathbf{S} &= \int_{-1}^{1}\int_{-1}^{1} \langle -x^2, xy + 2xy, y + x\rangle \cdot \langle 0, 1, 1\rangle\; dx\; dy \\ &= \int_{-1}^{1}\int_{-1}^{1} y + x\; dx\; dy \\ &= \boxed{0}.\end{aligned}$$

9. By Stokes' theorem, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S},$$

where S is the portion of the paraboloid in the first octant, oriented upward. The surface is parametrized by x and y satisfying $x, y \geq 0$ and $x^2 + y^2 \leq 1$, and the normal at (x, y, z) is just the negative of the gradient, namely, $\langle 2x, 2y, 1\rangle$. Now

$$\text{curl}\,\mathbf{F} = \langle -y, -z, -x\rangle,$$

and so

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{\sqrt{1-x^2}} \langle -y, x^2+y^2-1, -x\rangle \cdot \langle 2x, 2y, 1\rangle \, dy \, dx$$
$$= \int_0^1 \int_0^{\sqrt{1-x^2}} -2xy + 2x^2y + 2y^3 - 2y - x \, dy \, dx$$
$$= \int_0^{\pi/2} \int_0^1 -2r^3 \cos\theta \sin\theta + 2r^4 \cos^2\theta \sin\theta + 2r^4 \sin^3\theta - 2r^2 \sin\theta - r^2 \cos\theta \, dr \, d\theta$$
$$= \int_0^{\pi/2} -\frac{1}{2}\cos\theta\sin\theta + \frac{2}{5}\cos^2\theta\sin\theta + \frac{2}{5}\sin^3\theta - \frac{2}{3}\sin\theta - \frac{1}{3}\cos\theta \, d\theta.$$

After writing $\cos\theta\sin\theta$ as $\frac{1}{2}\sin 2\theta$ and $\cos^2\sin\theta + \sin^3\theta$ as $\sin\theta$, this becomes

$$\int_0^{\pi/2} -\frac{1}{4}\sin 2\theta + \frac{2}{5}\sin\theta - \frac{2}{3}\sin\theta - \frac{1}{3}\cos\theta \, d\theta = -\frac{1}{4} + \frac{2}{3} - \frac{2}{5} - \frac{1}{3}$$
$$= \frac{-15+24-40-20}{60}$$
$$= -\frac{-51}{60}$$
$$= \boxed{-\frac{17}{20}}.$$

15. Let C be the boundary of S, with the orientation induced by that of S. We can parametrize C via $\mathbf{r}(t) = \langle \cos t, 0, -\sin t\rangle$, and then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 0, -\sin t, \cos t\rangle \cdot \langle -\sin t, 0, -\cos t\rangle \, dt$$
$$= \int_0^{2\pi} -\cos^2 t \, dt$$
$$= -\pi.$$

We now evaluate the surface integral over S. The hemisphere can be parametrized as $\mathbf{r}(x,z) = \langle x, \sqrt{1-x^2-z^2}, z\rangle$, and then the normal is just the negative of the gradient, which is $\left\langle \frac{x}{\sqrt{1-x^2-z^2}}, 1, \frac{z}{\sqrt{1-x^2-z^2}} \right\rangle$. Combining this with the fact that curl $\mathbf{F} = \langle -1, -1, -1\rangle$, this gives

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \langle -1,-1,-1\rangle \cdot \left\langle \frac{x}{\sqrt{1-x^2-z^2}}, 1, \frac{z}{\sqrt{1-x^2-z^2}} \right\rangle \, dz \, dx.$$

Now since $x/\sqrt{1-x^2-z^2}$ changes signs after replacing x by $-x$, the integral will be 0. Similarly, $z/\sqrt{1-x^2-z^2}$ will integrate to 0, so we just get

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -1 \, dz \, dx$$
$$= -\pi.$$

Since this equals the line integral over C, Stokes' theorem holds in this case.

16. Let S be the surface in the plane $x + y + z = 1$ with boundary C. The curl of $\langle z, -2x, 3y \rangle$ is $\langle 3, 1, -2 \rangle$, and the normal to the plane is $\langle 1, 1, 1 \rangle$, so by Stokes' theorem, we have

$$\begin{aligned}\int_C z\,dx - 2x\,dy + 3y\,dz &= \int_S \langle 3, 1, -2 \rangle \cdot \langle 1, 1, 1 \rangle \,dA \\ &= \int_S 2\,dA,\end{aligned}$$

which is just twice the surface area of S. Thus, the integral depends only on the area enclosed by C.

17. To find the work done, we need to compute the line integral over the parallelogram with vertices $(0, 0, 0), (1, 0, 0), (1, 2, 1)$, and $(0, 2, 1)$. By Stokes' theorem, this is the same as the surface integral of curl $\mathbf{F}$ over the parallelogram. Since the vectors $\langle 1, 0, 0 \rangle$ and $\langle 0, 2, 1 \rangle$ span the parallelogram, we can parametrize it via

$$\mathbf{r}(u, v) = u\langle 1, 0, 0 \rangle + v\langle 0, 2, 1 \rangle = \langle u, 2v, v \rangle,$$

where $0 \le u, v \le 1$. Taking the cross product gives

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -1, 2 \rangle.$$

Finally, curl $\mathbf{F} = \langle 8y, 2z, 2y \rangle$, so putting it all together, we get

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^1 \langle 16v, 2v, 4v \rangle \cdot \langle 0, -1, 2 \rangle \,du\,dv \\ &= \int_0^1 \int_0^1 6v\,du\,dv \\ &= \boxed{3}.\end{aligned}$$

19. By Stokes' theorem, the surface integral of curl $\mathbf{F}$ equals the line integral of $\mathbf{F}$ over the boundary of S. But S has no boundary, so the line integral over the boundary is just 0.