Section 16.7

Problem 25.

Proof. We compute \( \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \) for \( \mathbf{F} = (x, y, z^2) \) for \( S \) the unit sphere, with outward normal \( \mathbf{n} = (x, y, z) \). Well, \( \mathbf{F} \cdot \mathbf{n} = x^2 + y^2 + z^3 \), and when we integrate over the sphere, the \( z^3 \) term will drop out since it is odd. So we are left with computing \( \iint_S (x^2 + y^2) \, dS \). We have \( \iint_S (x^2 + y^2) \, dS = 2\pi \cdot \int_0^\pi \sin^3 \phi \, d\phi = \frac{8\pi}{3} \) (an integral we’ve probably computed before; decompose the integrand as \( \sin \phi (1 - \cos^2 \phi) \) and use the chain rule). We can check the answer by using the divergence theorem.

Problem 30.

Proof. We compute \( \iint_S \mathbf{F} \cdot dS \) for \( \mathbf{F} = (x, y, 5) \) and \( S \) the region between \( x^2 + z^2 = 1 \) and the planes \( y = 0 \) and \( x + y = 2 \). We always take outward-pointing normals for surfaces without boundary. The surface has three parts: \( S_1, S_2, S_3 \), which are the \( y = 0 \) disk, the cylindrical side, and the flat face with an oval boundary, respectively. We calculate the separate fluxes and then take their sum.

First for \( S_1 \): the outward unit normal is \( (0, -1, 0) \), so \( \mathbf{F} \cdot \mathbf{n} = -y \). But \( y = 0 \) along \( S_1 \), so this is the 0 vector field, so the flux contribution of \( S_1 \) is zero.

For \( S_2 \), the outward unit normal is \( \mathbf{n} = (x, 0, z) \), so \( \mathbf{F} \cdot \mathbf{n} = x^2 + 5z \). Since \( S_2 \) is symmetric under the reflection \( z \mapsto -z \), we see that the \( 5z \) term in the dot product will contribute nothing to the flux. So we are left with evaluating \( \iint_{S_2} x^2 \, dS \). We can parametrize the two halves of \( S_2 \) by \( \mathbf{r}_\pm(x, y) = (x, y, \pm\sqrt{1-x^2}) \), where we have \( x \in [-1, 1] \) and \( y \) varying from 0 to \( 2 - x \). The Jacobian is \( \frac{1}{\sqrt{1-x^2}} \), and since the contribution from the upper and lower parts of \( S_2 \) are the same, we have that the flux through \( S_2 \) is
\[
2 \int_{-1}^{1} \int_{0}^{2-x} \frac{x^2}{\sqrt{1-x^2}} \, dy \, dx = 2 \int_{-1}^{1} \frac{2x^2 - x^3}{\sqrt{1-x^2}} \, dx.
\]
Since we integrate over \([-1, 1]\), the odd \( x^3 \) term drops out, and we are left with four times \( \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} \, dx \).

We can calculate this by trig substitution: \( u = \sin x \) will work. You can check that in the end, the integral will be
\[
\frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} \bigg|_{-1}^{1} = \frac{1}{2} (\pi/2 - (-\pi/2) - 0) = \frac{\pi}{2},
\]
which is 2\( \pi \) when multiplied by 4.

For \( S_3 \), the outward unit normal is \( \frac{1}{\sqrt{2}}(1, 1, 0) \), so \( \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{2}}(x + y) \). But along \( S_3 \), \( x + y = 2 \), so this dot product is just \( \sqrt{2} \). We now need to integrate it over \( S_3 \), which we can think of as the region in
the plane bounded by an ellipse with a semiminor axis of length 1 and a semimajor axis of length $\sqrt{2}$; thus the flux is $\sqrt{2} \times (\pi \cdot 1 \cdot \sqrt{2}) = 2\pi$.

Thus our total flux is the sum of all these three contributions: $0 + 2\pi + 2\pi = 4\pi$.

Problem 38.

Proof. If $x = k(y, z)$ for a function $k$, then we take as our parametrization $r(y, z) = (k(y, z), y, z)$; thus

$$\mathbf{F} \cdot (\mathbf{r}_y \times \mathbf{r}_z) = (P, Q, R) \cdot \begin{vmatrix} i & j & k \\ ky & 1 & 0 \\ kz & 0 & 1 \end{vmatrix} = (P, Q, R) \cdot (1, -ky, -kz) = P - kyQ - kzR$$

and so the expression for the flux follows.

Section 16.9

Problem 3.

Proof. We calculate both the flux and the integral of the divergence of $\mathbf{F} = (z, y, x)$. The divergence of $\mathbf{F}$ is 1, so $\iiint_E \text{div} \mathbf{F} \, dV = \frac{4}{3} \pi \cdot 4^3$ is the volume of the ball $E$.

To calculate the flux $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$, we need to know that the unit normal on $S$ can be written as $\mathbf{n} = \frac{1}{7}(x, y, z)$, and so $\mathbf{F} \cdot \mathbf{n} = \frac{1}{7}y^2 + \frac{1}{2}xz$. We now set up our double integral in spherical coordinates:

$$\iint_S \frac{1}{4}y^2 + \frac{1}{2}xz \, dS = \int_0^{2\pi} \int_0^\pi \left(\frac{1}{4}r^2 \sin^2 \phi \sin^2 \theta + \frac{1}{2}r^2 \sin \phi \cos \phi \cos \theta\right) r^2 \sin \phi \, d\phi \, d\theta$$

$$= \frac{4^4}{3} \int_0^{2\pi} \left(\frac{1}{4} \sin^2 \theta + \frac{1}{3} \sin \phi \cos \phi \right) \, d\theta$$

where in the second equality we used that $\int_0^\pi \sin^3 \theta \, d\theta = \frac{4}{3}$ and that $\sin^2 \phi \cos \phi$ is odd about $\pi/2$, so integrates to 0 over $[0, \pi]$. In the last equality we used that $\int_0^{2\pi} \sin^2 \theta \, d\theta = \pi$. So the two integrals agree.

Problem 7.

Proof. We first find the divergence:

$$\text{div} \mathbf{F} = \text{div} (3xy^2, xe^z, z^3) = 3y^2 + 0 + 3z^2.$$ 

Then we integrate this over our region, changing to polar coordinates in the $y, z$ plane:

$$\iint_E \text{div} \mathbf{F} \, dV = \int_{-1}^{1} \int_0^{2\pi} \int_0^1 3r^2 \, r \, rd\theta \, dx$$

$$= 3 \cdot 2\pi \cdot \frac{3}{4} \cdot \frac{9\pi}{2} = \frac{9\pi}{2}$$

By the divergence theorem, this is also our flux integral.
Problem 9.

**Proof.** The divergence of \( \mathbf{F} \) is

\[
div (xe^y, z - e^y, -xy) = e^y - e^y + 0 = 0.
\]

So the flux is zero.

\[
\square
\]

Problem 14.

**Proof.** For \( \mathbf{r} = (x, y, z) \), we are given \( \mathbf{F} = |\mathbf{r}|^2 \mathbf{r} = (x^2 + y^2 + z^2)(x, y, z) \). The divergence is

\[
(3x^2 + y^2 + z^2) + (3y^2 + x^2 + z^2) + (3z^2 + x^2 + y^2) = 5(x^2 + y^2 + z^2)
\]

So the integral over the ball of radius \( R \) centered at 0 is

\[
\iiint_E \text{div} \mathbf{F} \, dV = 5 \int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 20\pi \cdot \frac{1}{5} R^5 = 4\pi R^5.
\]

\[
\square
\]

Problem 17.

**Proof.** Our vector field is \( \mathbf{F} = (z^2x, \frac{1}{3}y^3 + \tan z, x^2z + y^2) \), and we wish to compute the flux through the surface \( S \) which is the upper unit hemisphere centered at 0. We can’t apply the divergence theorem directly because this isn’t a closed surface, but we can cap it off with the unit disk \( D \) in the \( xy \) plane to get a closed surface \( S_1 \) and apply the divergence theorem to that. After, we will simply subtract the flux through the disk part to get the flux through our original \( S \).

To that end, let \( E \) denote the region enclosed by \( S_1 \). Then the divergence integral is

\[
\iiint_E \text{div} \mathbf{F} \, dV = \int_0^1 \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \rho^2 \rho^2 \sin \phi d\phi \, d\theta \, d\rho
\]

\[
= \frac{2\pi}{5}
\]

Now, the flux through the disk \( D \) (with outward-pointing normal vector \( \mathbf{n} = (0, 0, -1) \)) is

\[
\iint_D \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D (-y^2) \, dS
\]

\[
= \int_0^{2\pi} \int_0^1 -r^2 \sin^2 \theta \, r \, dr \, d\theta
\]

\[
= -\frac{\pi}{4}
\]

The flux through just \( S \) is the difference of the two:

\[
\frac{2\pi}{5} - \frac{\pi}{4} = \frac{3\pi}{20}.
\]

\[
\square
\]
Problem 23.

Proof. We consider the electric field \( E = \frac{eQ}{|x|^3} x \), and show that its divergence is zero. Well, \( \frac{1}{|x|^3} = (x^2 + y^2 + z^2)^{-3/2} \). Let’s evaluate the first derivative of the first component, ignoring the constant:

\[
\frac{\partial}{\partial x} \left( \frac{1}{|x|^3} x \right) = \frac{-3x^2}{|x|^5} + \frac{1}{|x|^3}.
\]

When we add these partials for all variables, everything cancels.

\[\square\]

Problem 24.

Proof. The outward unit normal on a sphere of radius 1 is \( n = (x, y, z) \), and so we can realize this surface integral as a flux:

\[
\iint_S (2x + 2y + z^2) \, dS = \iiint_E (2, 2, z) \cdot n \, dS = \iiint_E 1 \, dV,
\]

where we used the divergence theorem in the last equality. So our flux is the same as the volume of the unit ball, which we know is \( \frac{4}{3} \pi \).

\[\square\]

Problem 27.

Proof. Assume \( S \) is a surface without boundary, bounding the region \( E \), and that together they satisfy the conditions of the divergence theorem. Then the divergence theorem says that

\[
\iint_S \nabla \times F \cdot n \, dS = \iiint_E \nabla \cdot \nabla \times F \, dV,
\]

but the divergence of a curl is always zero, so this integral vanishes.

\[\square\]

Problem 31.

Proof. Let \( f \) be a scalar functions, and \( S, E \) satisfy the conditions of the divergence theorem. Consider a vector field \( F = fc \) for \( c \) a constant vector. The hint suggests that we take the divergence of this vector field, which is \( \nabla \cdot F = \nabla f \cdot c \). Using this, let’s apply the divergence theorem to \( F \):

\[
\iint_S F \cdot n \, dS = \iint_S fc \cdot n \, dS = \left( \iint_S fn \, dS \right) \cdot c = \iiint_E \nabla \cdot F \, dV = \left( \iiint_E \nabla f \, dV \right) \cdot c,
\]

where we have moved \( c \) outside the integrals. These equalities are true for arbitrary \( c \), since we were free to choose any \( c \). The integrals are vectors, so by varying \( c \) across the vectors \((1, 0, 0), (0, 1, 0), (0, 0, 1)\), we recover equality of their components. Thus \( \iint_S f n \, dS = \iiint_E \nabla f \, dV \).

\[\square\]