Math 53 Homework 13, Part II

due Wednesday, April 25

Section 16.7

Problem 25.

Proof. We compute $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ for $\mathbf{F} = (x, y, z^2)$ for S the unit sphere, with outward normal $\mathbf{n} = (x, y, z)$. Well, $\mathbf{F}\cdot\mathbf{n} = x^2 + y^2 + z^3$, and when we integrate over the sphere, the z^3 term will drop out since it is odd. So we are left with computing

$$\iint_S (x^2+y^2)\,dS = \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\,\sin\phi\,d\phi\,d\theta = 2\pi\cdot\int_0^{\pi}\sin^3\phi\,d\phi = \frac{8\pi}{3}$$

(an integral we've probably computed before; decompose the integrand as $\sin\phi(1-\cos^2\phi)$ and use the chain rule). We can check the answer by using the divergence theorem. □

Problem 30.

Proof. We compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for $\mathbf{F} = (x, y, 5)$ and S the region between $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$. We always take outward-pointing normals for surfaces without boundary. The surface has three parts: S_1, S_2, S_3, which are the $y = 0$ disk, the cylindrical side, and the flat face with an oval boundary, respectively. We calculate the separate fluxes and then take their sum.

First for S_1: the outward unit normal is $(0, -1, 0)$, so $\mathbf{F}\cdot\mathbf{n} = -y$. But $y = 0$ along S_1, so this is the 0 vector field, so the flux contribution of S_1 is zero.

For S_2, the outward unit normal is $\mathbf{n} = (x, 0, z)$, so $\mathbf{F}\cdot\mathbf{n} = x^2 + 5z$. Since S_2 is symmetric under the reflection $z \mapsto -z$, we see that the $5z$ term in the dot product will contribute nothing to the flux. So we are left with evaluating $\iint_{S_2} x^2\,dS_2$. We can parametrize the two halves of S_2 by $\mathbf{r}_\pm(x, y) = (x, y, \pm\sqrt{1-x^2})$, where we have $x \in [-1, 1]$ and y varying from 0 to $2-x$. The Jacobian is $\frac{1}{\sqrt{1-x^2}}$, and since the contribution from the upper and lower parts of S_2 are the same, we have that the flux through S_2 is

$$2\int_{-1}^{1}\int_0^{2-x}\frac{x^2}{\sqrt{1-x^2}}\,dy\,dx = 2\int_{-1}^{1}\frac{2x^2-x^3}{\sqrt{1-x^2}}\,dx.$$

Since we integrate over $[-1, 1]$, the odd x^3 term drops out, and we are left with four times $\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}\,dx$. We can calculate this by trig substitution: $u = \sin x$ will work. You can check that in the end, the integral will be

$$\frac{1}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2}\Big|_{-1}^{1} = \frac{1}{2}(\pi/2 - (-\pi/2) - 0) = \frac{\pi}{2},$$

which is 2π when multiplied by 4.

For S_3, the outward unit normal is $\frac{1}{\sqrt{2}}(1, 1, 0)$, so $\mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{2}}(x+y)$. But along S_3, $x + y = 2$, so this dot product is just $\sqrt{2}$. We now need to integrate it over S_3, which we can think of as the region in the plane bounded by an ellipse with a semiminor axis of length 1 and a semimajor axis of length $\sqrt{2}$; thus the flux is $\sqrt{2} \times (\pi \cdot 1 \cdot \sqrt{2}) = 2\pi$.

Thus our total flux is the sum of all these three contributions: $0 + 2\pi + 2\pi = 4\pi$. □

Problem 38.

Proof. If $x = k(y, z)$ for a function k, then we take as our parametrization $\mathbf{r}(y, z) = (k(y, z), y, z)$; thus

$$\begin{aligned}
\mathbf{F} \cdot (\mathbf{r}_y \times \mathbf{r}_z) &= (P, Q, R) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ k_y & 1 & 0 \\ k_z & 0 & 1 \end{vmatrix} \\
&= (P, Q, R) \cdot (1, -k_y, -k_z) \\
&= P - k_y Q - k_z R
\end{aligned}$$

and so the expression for the flux follows. □

Section 16.9

Problem 3.

Proof. We calculate both the flux and the integral of the divergence of $\mathbf{F} = (z, y, x)$. The divergence of $\mathbf{F}$ is 1, so $\iiint_E \operatorname{div} \mathbf{F}\, dV = \frac{4}{3}\pi \cdot 4^3$ is the volume of the ball E.

To calculate the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, we need to know that the unit normal on S can be written as $\mathbf{n} = \frac{1}{4}(x, y, z)$, and so $\mathbf{F} \cdot \mathbf{n} = \frac{1}{4}y^2 + \frac{1}{2}xz$. We now set up our double integral in spherical coordinates:

$$\begin{aligned}
\iint_S \frac{1}{4}y^2 + \frac{1}{2}xz\, dS &= \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{4} 4^2 \sin^2 \phi \sin^2 \theta + \frac{1}{2} 4^2 \sin \phi \cos \phi \cos \theta \right) 4^2 \sin \phi\, d\phi\, d\theta \\
&= 4^4 \int_0^{2\pi} \left(\frac{1}{4} \frac{4}{3} \sin^2 \theta + 0 \right) d\theta \\
&= \frac{4^4 \cdot \pi}{3}
\end{aligned}$$

where in the second equality we used that $\int_0^{\pi} \sin^3 \theta\, d\theta = \frac{4}{3}$ and that $\sin^2 \phi \cos \phi$ is odd about $\pi/2$, so integrates to 0 over $[0, \pi]$. In the last equality we used that $\int_0^{2\pi} \sin^2 \theta\, d\theta = \pi$. So the two integrals agree. □

Problem 7.

Proof. We first find the divergence:

$$\operatorname{div} \mathbf{F} = \operatorname{div} (3xy^2, xe^z, z^3) = 3y^2 + 0 + 3z^2.$$

Then we integrate this over our region, changing to polar coordinates in the y, z plane:

$$\begin{aligned}
\iint_E \operatorname{div} \mathbf{F}\, dV &= \int_{-1}^{2} \int_0^{2\pi} \int_0^1 3r^2\, r\, dr\, r\theta\, dx \\
&= 3 \cdot 2\pi \cdot \frac{3}{4} = \frac{9\pi}{2}
\end{aligned}$$

By the divergence theorem, this is also our flux integral. □

Problem 9.

Proof. The divergence of **F** is

$$\text{div}\,(xe^y, z - e^y, -xy) = e^y - e^y + 0 = 0.$$

So the flux is zero. □

Problem 14.

Proof. For $\mathbf{r} = (x, y, z)$, we are given $\mathbf{F} = |\mathbf{r}|^2\mathbf{r} = (x^2 + y^2 + z^2)(x, y, z)$. The divergence is

$$(3x^2 + y^2 + z^2) + (3y^2 + x^2 + z^2) + (3z^2 + x^2 + y^2) = 5(x^2 + y^2 + z^2)$$

So the integral over the ball of radius R centered at 0 is

$$\begin{aligned}\iiint_E \text{div}\,\mathbf{F}\,dV &= 5\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \\ &= 20\pi \cdot \frac{1}{5}R^5 = 4\pi R^5.\end{aligned}$$

□

Problem 17.

Proof. Our vector field is $\mathbf{F} = (z^2x, \frac{1}{3}y^3 + \tan z, x^2z + y^2)$, and we wish to compute the flux through the surface S which is the upper unit hemisphere centered at 0. We can't apply the divergence theorem directly because this isn't a closed surface, but we can cap it off with the unit disk D in the xy plane to get a closed surface S_1 and apply the divergence theorem to that. After, we will simply subtract the flux through the disk part to get the flux through our original S.

To that end, let E denote the region enclosed by S_1. Then the divergence integral is

$$\begin{aligned}\iiint_E \text{div}\,\mathbf{F}\,dV &= \iiint_E z^2 + y^2 + x^2\,dV \\ &= \int_0^1\int_0^{2\pi}\int_0^{\pi/2} \rho^2\rho^2 \sin\phi d\phi\,d\theta\,d\rho \\ &= \frac{2\pi}{5}\end{aligned}$$

Now, the flux through the disk D (with outward-pointing normal vector $\mathbf{n} = (0, 0, -1)$) is

$$\begin{aligned}\iint_D \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_D (-y^2)\,dS \\ &= \int_0^{2\pi}\int_0^1 -r^2\sin^2\theta\,r\,dr\,d\theta \\ &= -\frac{\pi}{4}\end{aligned}$$

The flux through just S is the difference of the two:

$$\frac{2\pi}{5} - -\frac{\pi}{4} = \frac{3\pi}{20}.$$

□

Problem 23.

Proof. We consider the electric field $\mathbf{E} = \frac{\epsilon Q}{|\mathbf{x}|^3}\mathbf{x}$, and show that its divergence is zero. Well, $\frac{1}{|\mathbf{x}|^3} = (x^2 + y^2 + z^2)^{-3/2}$. Let's evaluate the first derivative of the first component, ignoring the constant:

$$\frac{\partial}{\partial x}\left(\frac{1}{|\mathbf{x}|^3}x\right) = \frac{-3x^2}{|\mathbf{x}|^5} + \frac{1}{|\mathbf{x}|^3}.$$

When we add these partials for all variables, everything cancels.

□

Problem 24.

Proof. The outward unit normal on a sphere of radius 1 is $\mathbf{n} = (x, y, z)$, and so we can realize this surface integral as a flux:

$$\iint_S (2x + 2y + z^2)\, dS = \iint_S (2, 2, z) \cdot \mathbf{n}\, dS = \iiint_E 1\, dV,$$

where we used the divergence theorem in the last equality. So our flux is the same as the volume of the unit ball, which we know is $\frac{4}{3}\pi$. □

Problem 27.

Proof. Assume S is a surface without boundary, bounding the region E, and that together they satisfy the conditions of the divergence theorem. Then the divergence theorem says that

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \nabla \times \mathbf{F}\, dV,$$

but the divergence of a curl is always zero, so this integral vanishes. □

Problem 31.

Proof. Let f be a scalar functions, and S, E satisfy the conditions of the divergence theorem. Consider a vector field $\mathbf{F} = f\mathbf{c}$ for $\mathbf{c}$ a constant vector. The hint suggests that we take the divergence of this vector field, which is $\nabla \cdot \mathbf{F} = \nabla f \cdot \mathbf{c}$. Using this, let's apply the divergence theorem to $\mathbf{F}$:

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S f\mathbf{c} \cdot \mathbf{n}\, dS &= \left(\iint_S f\mathbf{n}\, dS\right) \cdot \mathbf{c} \\ &= \iiint_E \nabla \cdot \mathbf{F}\, dV \\ &= \left(\iiint_E \nabla f\, dV\right) \cdot \mathbf{c},\end{aligned}$$

where we have moved $\mathbf{c}$ outside the integrals. These equalities are true for arbitrary $\mathbf{c}$, since we were free to choose any $\mathbf{c}$. The integrals are vectors, so by varying $\mathbf{c}$ across the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, we recover equality of their components. Thus $\iint_S f\mathbf{n}\, dS = \iiint_E \nabla f\, dV$. □