Solutions to Homework 12-Part 1

16.6

Problem 41. First, we parametrize the surface by treating it as a graph of the function$z = 1 - \frac{1}{3}x - \frac{2}{3}y$ over $x^2 + y^2 \leq 3$. Therefore $dS = \sqrt{1 + z_x^2 + z_y^2}dxdy = \frac{\sqrt{14}}{3}dxdy$ and the area $A = \iint_{x^2+y^2\leq 3} \frac{\sqrt{14}}{3}dxdy = \sqrt{14}\pi$. □

Problem 45. First, we parametrize the surface by treating it as a graph of the function$z = xy$ over $x^2+y^2 \leq 1$. Therefore $dS = \sqrt{1 + z_x^2 + z_y^2}dxdy = \sqrt{1 + x^2 + y^2}dxdy$ and the area $A = \iint_{x^2+y^2\leq 1} \sqrt{1 + x^2 + y^2}dxdy = \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}rdrd\theta = \int_0^{2\pi} (\frac{1}{3}\sqrt{1 + r^2}^3)|_0^1 d\theta = \int_0^{2\pi} \frac{\sqrt{8}-1}{3}d\theta = 2\pi\frac{\sqrt{8}-1}{3}$. □

Problem 50. There are two components in the surface, since the two components are symmetric, we will only compute the area for the top cap, then multiply it by 2.

To parametrize the surface, we will use the sphereical coordinates, hence $x = b \sin\phi \cos\theta, y = b \sin\phi \sin\theta, z = b \cos\phi$, where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \arcsin\frac{a}{b}$. Since we know the area form dS in sphereical coordinates is $b^2 \sin\phi d\theta \dagger \phi$, we need to evaluate the following integeral:

$$\int_0^{2\pi} \int_0^{\arcsin\frac{a}{b}} b^s \sin\phi d\phi\theta = 2\pi b^2(1 - \cos\arcsin\frac{a}{b}) = 2\pi b^2(1 - \frac{\sqrt{b^2 - a^2}}{b}) = 2\pi b^2 - 2\pi b\sqrt{b^2 - a^2}$$

So the total area is $4\pi b^2 - 4\pi b\sqrt{b^2 - a^2}$. □

Problem 61. Given $x^2 + y^2 + z^2 = 4z$ and $z = x^2 + y^2$, we know that $z + z^2 = 4z$, hence $z = 0, 3$. Therefore the part of sphere in inside $z = x^+y^2$ is $x^2 + y^2 + z^2 = 4z$ and $z \geq 3$. If we move the sphere's center back to the origin, the surface is $x^2 + y^2 + z^2 = 4$ and $z \geq 1$. Therefore the surface area can be computed using sphereical coordinates:

$$A = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} 4 \sin\phi d\phi d\theta = 8\pi(1 - \frac{\sqrt{3}}{2}).$$

□

Problem 62. There are four identical pieces of surface, we can compute the area of oue and then multiply it by 4. To that end, let's parametrize the blue surface in the textbook's picture. That surface is inside $x^2 + z^2 = 1$, we can view x as a function of y, z. Hence the parametrization is $x = \sqrt{1 - z^2}, y = y, z = z$. To find the domain of y, z, we can find the intersection of the surfaces and project the intersectiom curves to the yz plane. Such projected curve is $y^2 + z^2 = 1$, hence the domain for y, z is the unit disk. Therefore the area can be computed by

$$\iint_{y^2+z^2\leq 1} \sqrt{1 + x_y^2 + x_z^2}dydz = \iint_{y^2+z^2\leq 1} \frac{1}{\sqrt{1 - z^2}}dydz = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}dydz = \int_{-1}^{1} 2dz = 4$$

The total area is 16. □

Problem 64. (a).

$$\boldsymbol{r}(\theta,\alpha) = \langle b\cos\theta, b\sin\theta, 0\rangle + \langle a\cos\theta\cos\alpha, a\sin\theta\cos\alpha, a\sin\alpha\rangle$$

$0 \le \theta \le 2\pi$ and $0 \le \alpha \le 2\pi$.

(c).

$$\boldsymbol{r}_\theta = \langle -b\sin\theta - a\sin\theta\cos\alpha, b\cos\theta + a\cos\theta\cos\alpha, 0\rangle$$

$$\boldsymbol{r}_\alpha = \langle -a\cos\theta\sin\alpha, -a\sin\theta\sin\alpha, a\cos\alpha\rangle$$

Then

$$\boldsymbol{r}_\theta \times \boldsymbol{r}_\alpha = \langle ab\cos\theta\cos\alpha + a^2\cos\theta\cos^2\alpha, ab\sin\theta\cos\alpha + a^2\sin\theta\cos^2\alpha, ab\sin\alpha + a^2\sin\alpha\cos\alpha\rangle$$

Therefore $|\boldsymbol{r}_\theta \times \boldsymbol{r}_\alpha| = a(b + a\cos\alpha)$. Hence the area is

$$\int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)d\alpha d\theta = 4\pi^2 ab$$

□

16.7

Problem 11. The surface can be though at the graph of the function $z = 4 - 4x + 2y$ for x, y in the triangle domain with vertices $(0,0), (1,0), (0,-2)$. Since $dS = \sqrt{1 + z_x^2 + z_y^2 dxdy} = \sqrt{21}dxdy$. Hence

$$\iint xdS = \int_0^1 \int 2x - 2^0 xdydx = \int_0^1 (2-2x)xdx = \frac{1}{3}$$

□

Problem 16. Using the sphereical coordinates to parametrize the surface, we have $y^2 = \sin\theta^2\sin^2\phi$. Since $dS = \sin\phi d\theta d\phi$, we have

$$\iint y^2 dS = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\theta^2 \sin\phi^3 d\phi d\theta = \int_0^{2\pi} \sin\theta^2 d\theta \int_0^{\frac{\pi}{4}} \sin\phi^3 d\phi$$

Since $\int_0^{2\pi} \sin\theta^2 d\theta = \int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta = \pi$ and $\int_0^{\frac{\pi}{4}} \sin\phi^3 d\phi = int_0^{\frac{\pi}{4}}(\cos\phi^2 - 1)d\cos\phi = \int_{\frac{\sqrt{2}}{2}}^1 (1-u^2)du = (1 - \frac{\sqrt{2}}{2}) - \frac{1}{3} + \frac{1}{3}\frac{\sqrt{2}}{4} = \frac{2}{3} - \frac{5\sqrt{2}}{12}$. Hence the final answer is$(\frac{2}{3} - \frac{5\sqrt{2}}{12})\pi$. □

Problem 19. Since $f(x,y,z) = xz$ has the property that $f(x,y,-z) = -f(x,y,z)$, and the surface is symetric with respect to the reflection through the xy plane, therefore $\iint_S xzdS = 0$. □

Problem 24. Treat the surface as a graph of function, since we are using the downward orientation, we have $d\boldsymbol{S} = \langle z_x, z_y, -1\langle dxdy = \langle \frac{x}{z}, \frac{y}{z}, -1\rangle dxdy$. Hence

$$\iint \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{1\le x^2+y^2\le 9} -\frac{x^2}{z} - \frac{y^2}{z} - z^3 dxdy = \iint_{1\le x^2+y^2\le 9} -z - z^3 dxdy = \int_0^{2\pi}\int_0^3 (-r - r^3)rdrd\theta = -\frac{1712}{15}\pi$$

□