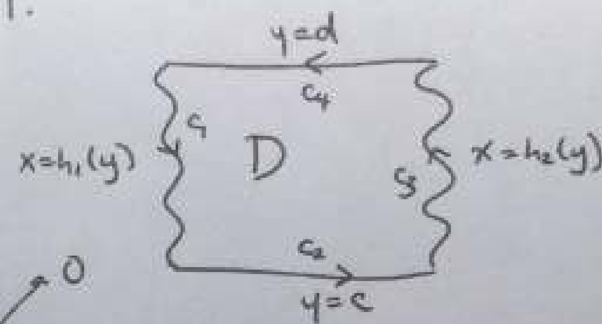
First, we prove ⑥ for type II regions, or regions

$$D = \{ h_1(y) \le x \le h_2(y), \quad c \le y \le d \}.$$



Then

$$\int_C Q\,dy$$

$$= \int_{C_1} Q\,dy + \cancelto{0}{\int_{C_2} Q\,dy} + \int_{C_3} Q\,dy + \cancelto{0}{\int_{C_4} Q\,dy}$$

$$= \int_d^c Q(h_1(y), y)\,dy + \int_c^d Q(h_2(y), y)\,dy$$

$$= \int_c^d Q(h_2(y), y) - Q(h_1(y), y)\,dy$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} Q_x \,dx\,dy.$$

To see that ⑥ is true for general regions, we cut it into type II regions. By adding the line integrals over the boundary of these regions, we obtain

$$\int_C Q\,dy = \iint_R Q_x\,dA.$$