Math 53 Problem Set 1 Due Jan 24th, 2018

Section 12.1

3) *Which of the points* $A(-4, 0, -1)$, $B(3, 1, -5)$, *and* $C(2, 4, 6)$ *is closest to the* yz*-plane? Which point lies in the* xz*-plane?*

Solution: The yz-plane is where $x = 0$, so the point with the smallest x component will be the point closest to the yz-plane. Hence, C is the closest point to the yz-plane

The xz-plane is where $y = 0$, so A lies in the xz-plane.

19) *Show the equation represents a sphere, and find its center and radius.*

$$2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$$

Solution: The general equation for a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

To put our equation into this form, move $8x - 24z$ to the other side of the equation and complete the squares.

$$\begin{aligned}
2x^2 + 2y^2 + 2z^2 &= 8x - 24z + 1 \\
2x^2 - 8x + 2y^2 + 2z^2 + 24z &= 1 \\
2(x^2 - 4x + 4) + 2y^2 + 2(z^2 + 12z + 36) &= 1 + 2(4) + 2(36) \\
(x - 2)^2 + y^2 + (z + 6)^2 &= \frac{81}{2}
\end{aligned}$$

Therefore, this equation does describe a sphere with center $C(2, 0, -6)$ and radius $r = \frac{9}{\sqrt{2}}$.

21) *(a) Prove that the midpoint of the line segment from* $P_1(x_1, y_1, z_1)$ *to* $P_2(x_2, y_2, z_2)$ *is*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Solution: First, call the midpoint, M and the origin, O. The vector pointing from P_1 to P_2 is

$$\vec{P_1P_2} = \langle x_2 - x_1, y_2 - y_2, z_2 - z_1 \rangle$$

Then, the midpoint between P_1 and P_2 is given by $\vec{OP_1} + \frac{1}{2}\vec{P_1P_2}$. Hence,

$$\begin{aligned}
\vec{OM} &= \vec{OP_1} + \frac{1}{2}\vec{P_1P_2} \\
&= \langle x_1, y_1, z_1 \rangle + \frac{1}{2}\langle x_2 - x_1, y_2 - y_2, z_2 - z_1 \rangle \\
&= \left\langle \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right\rangle
\end{aligned}$$

Alternatively, notice that $\vec{P_1M} = \vec{MP_2}$.

$$\begin{aligned}
\vec{P_1M} &= \vec{OM} - \vec{OP_1} \\
&= \left\langle \frac{x_1 + x_2}{2} - x_1, \frac{y_1 + y_2}{2} - y_1, \frac{z_1 + z_2}{2} - z_1 \right\rangle \\
&= \left\langle \frac{x_2 - x_1}{2}, \frac{y_2 - y_1}{2}, \frac{z_2 - z_1}{2} \right\rangle \\
&= \left\langle x_2 - \frac{x_1 + x_2}{2}, y_2 - \frac{y_1 + y_2}{2}, z_2 - \frac{z_1 + z_2}{2} \right\rangle \\
&= \vec{OP_2} - \vec{OM} \\
&= \vec{MP_2}
\end{aligned}$$

(b) Find the length of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$.

Solution: First, calculate the midpoints using the above formula. Call the midpoint between A and B, M_{AB}. Then

$$M_{AB} = \left(-\frac{1}{2}, 1, 4\right)$$

Call the midpoint between B and C, M_{BC}. Then

$$M_{BC} = \left(1, \frac{1}{2}, 5\right)$$

Call the midpoint between A and C, M_{AC}. Then

$$M_{AC} = \left(\frac{5}{2}, \frac{3}{2}, 4\right)$$

Second, use the distance formula

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Hence, the distance between A and M_{BC} is

$$\begin{aligned} |AM_{BC}| &= \sqrt{(1-1)^2 + \left(\frac{1}{2} - 2\right)^2 + (5-3)^2} \\ &= \sqrt{0 + \frac{9}{4} + 4} \\ &= \sqrt{\frac{25}{4}} \\ &= \frac{5}{2} \end{aligned}$$

The distance between B and M_{AC} is

$$\begin{aligned} |BM_{AC}| &= \sqrt{\left(\frac{5}{2} + 2\right)^2 + \left(\frac{3}{2} - 0\right)^2 + (4-5)^2} \\ &= \frac{1}{2}\sqrt{94} \end{aligned}$$

The distance between C and M_{AB} is

$$\begin{aligned} |CM_{AB}| &= \sqrt{\left(-\frac{1}{2} - 4\right)^2 + (1-1)^2 + (4-5)^2} \\ &= \frac{1}{2}\sqrt{85} \end{aligned}$$

25) *Describe in words the region of* $\mathbb{R}^3$ *represented by* $x = 5$.

Solution: The plane parallel to the yz-plane going through $(5, 0, 0)$. Alternatively, the yz-plane shifted 5 units along the x-axis.

29) *Describe in words the region of* $\mathbb{R}^3$ *represented by* $0 \le z \le 6$.

Solution: The infinite volume enclosed by the the xy-plane and the plane parallel to the xy-plane that goes though $(0, 0, 6)$ including both of those planes.

Section 12.2 2) *What is the relationship between the point* $(4, 7)$ *and the vector* $\langle 4, 7\rangle$*? Illustrate with a sketch.*

Solution: The vector $\langle 4, 7\rangle$ points from the origin to the point $(4, 7)$.

3) *Name all the equal vectors in the parallelogram shown.*

Solution: $\vec{AD} = \vec{CB}$, $\vec{AB} = \vec{DC}$, $\vec{DE} = \vec{EB}$, and $\vec{CE} = \vec{EA}$. All of these vectors are parallel and have the same magnitude.

8) *If the vectors in the figure satisfy* $|\boldsymbol{u}| = |\boldsymbol{v}| = 1$ *and* $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w} = 0$, *what is* $|\boldsymbol{w}|$*?*

Solution: From the assumptions $\mathbf{u} + \mathbf{v} = -\mathbf{w}$, so

$$\begin{aligned} |\mathbf{w}|^2 &= |\mathbf{u} + \mathbf{v}|^2 \\ &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= 1 + 0 + 1 \end{aligned}$$

where the last line follows from the assumption $|\mathbf{u}| = |\mathbf{v}| = 1$ and because the drawing shows $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Therefore,

$$|\mathbf{w}| = \sqrt{2}$$

26) *Find the vector that has the same direction as* $\langle 6, 2, -3\rangle$ *but has length 4.*

Solution: First, find the unit vector that points in the same direction as $\langle 6, 2, -3\rangle$ and then multiply by 4. To calculate the unit vector, first, find the magnitude of $\langle 6, 2, -3\rangle$.

$$|\langle 6, 2, -3\rangle| = \sqrt{6^2 + 2^2 + (-3)^2} = sqrt49 = 7$$

Hence, $\frac{1}{7}\langle 6, 2, -3\rangle$ is a unit vector that points in the same direction as $\langle 6, 2, -3\rangle$. Thus, the vector we want is

$$\frac{4}{7}\langle 6, 2, -3\rangle$$

41) *Find the unit vector that are parallel to the tangent line to the parabola* $y = x^2$ *at the point* $(2, 4)$.

Solution: First, find the slope of the parabola $y = x^2$ at the point $(2, 4)$. The derivative equals

$$\frac{\partial y}{\partial x} = 2.$$

Hence, the slope of the tangent line is 4. Thus, the vector $\langle 1, 4\rangle$ is parallel to the tangent line. Then calculate the magnitude of $\langle 1, 4\rangle$.

$$|\langle 1, 4\rangle| = \sqrt{1^2 + 4^2} = \sqrt{17}$$

Therefore, the unit vector parallel to the tangent line is

$$\frac{1}{\sqrt{17}}\langle 1, 4\rangle$$

43) *If* A*,* B*, and* C *are the vertices of a triangle, find*

$$\vec{AB} + \vec{BC} + \vec{CA}$$

Solution: By direct calculation, $\vec{AB} + \vec{BC} = \vec{AC}$ and $\vec{CA} = -\vec{AC}$. Hence,

$$\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$$

You can see this visually because the vectors go all the way around the triangle, so the resulting vector must point from A to A, i.e. must be $\vec{0}$.

47) If $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$, describe the set of all points (x, y, z) such that $|\mathbf{r} - \mathbf{r}_0| = 1$.

Solution: Plug in the coordinates of the two vectors into the equation $|\mathbf{r} - \mathbf{r}_0|^2 = 1$.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$$

We recognize this as a sphere centered at (x_0, y_0, z_0) with radius 1.

Section 12.3

1) *Which of the following expressions are meaningful? Which are meaningless? Explain.*

(b) $(\boldsymbol{a} \cdot \boldsymbol{b})c$

This does have meaning. The dot product of two vectors, $\mathbf{a}\cdot\mathbf{b}$, gives a scalar. This scalar can multiply the vector $\mathbf{c}$.

(d) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$

This does have meaning. The expression can also be written $\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$).

(f) $|\boldsymbol{a}|\cdot(\boldsymbol{b}+\boldsymbol{c})$

This does NOT have meaning. The magnitude of a vector $|\mathbf{a}|$ is a scalar. You can only take the dot product of two vectors not a scalar and a vector.

11) If $\boldsymbol{u}$ *is a unit vector, find* $\boldsymbol{u}\cdot\boldsymbol{v}$ *and* $\boldsymbol{u}\cdot\boldsymbol{w}$*.*

Solution: Intuitively, the projection of $\mathbf{v}$ onto $\mathbf{u}$ is $\frac{1}{2}\mathbf{u}$, and the projection of $\mathbf{w}$ onto $\mathbf{u}$ is $-\frac{1}{2}\mathbf{u}$. Hence, $\mathbf{u}\cdot\mathbf{v}=\frac{1}{2}$ and $\mathbf{u}\cdot\mathbf{w}=-\frac{1}{2}$

Using the equation $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, this can be seen algebraically. Note, from the drawing that $|\mathbf{u}|=|\mathbf{v}|=\mathbf{w}=1$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{3}$. Hence,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\theta \\ &= \cos\left(\frac{\pi}{3}\right) \\ &= \frac{1}{2}\end{aligned}$$

The angle between $\mathbf{u}$ and $\mathbf{w}$ is $\frac{2\pi}{3}$. Hence,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{w} &= |\mathbf{u}||\mathbf{w}|\cos\theta \\ &= \cos\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{2}\end{aligned}$$

23) (a) Determine wether the given vectors are orthogonal, parallel, or neither. $\boldsymbol{a}=\langle 9,3\rangle$ *and* $\boldsymbol{b}=\langle -2,6\rangle$

Solution There is no scalar, c, such that $9c=-2$ and $3c=6$. Thus, the two vectors are not parallel.

To check if the two vectors are orthogonal, take the dot product.

$$\mathbf{a}\cdot\mathbf{b}=\langle 9,3\rangle\cdot\langle -2,6\rangle=-18+18=0$$

Therefore, these two vectors are orthogonal.

27) *Find a unit vector that is orthogonal to both* $\boldsymbol{i}+\boldsymbol{j}$ *and* $\boldsymbol{i}+\boldsymbol{k}$.

Solution: First, for a generic vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, the dot products are

$$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}) = a+b$$

$$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(\mathbf{i}+\mathbf{k}) = a+c$$

If we want to find a vector orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$, then set both of these equations to zero. Hence, $a=-b=-c$. Just the vector is of the form $a\mathbf{i}-a\mathbf{j}-a\mathbf{k}=a\,(\mathbf{i}-\mathbf{j}-\mathbf{k})$. Now take the magnitude of this vector.

$$|\mathbf{i}-\mathbf{j}-\mathbf{k}| = \sqrt{1^2+(-1)^2+(-1)^2} = \sqrt{3}$$

Therefore, let $a=\frac{1}{\sqrt{3}}$. Then the desired vector is

$$\frac{1}{\sqrt{3}}\,(\mathbf{i}-\mathbf{j}-\mathbf{k})$$

55) *Find the angle between a diagonal of a cube and one of its edges.*

Solution: Use the equation $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$. Take the unit cube in the first octant with edges $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Then take the dot product of the edge $\mathbf{i}$ and the diagonal $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

$$\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) = |\mathbf{i}||\mathbf{i}+\mathbf{j}+\mathbf{k}|\cos\theta$$

$$1 = 1\cdot\sqrt{3}\cos\theta$$

Hence, $\cos\theta = 1/\sqrt{3}$, or $\theta=\cos^{-1}(1/\sqrt{3})$.

63) *The Parallelogram Law states that*

$$|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2+2|\mathbf{a}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.

Notice, at $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are the two diagonals of the parallelogram. Hence, the Law states that the sum of the lengths of the two diagonals of a parallelogram is equal to the perimeter of the parallelogram.

(b) Prove the Parallelogram Law.

Use the hint.

$$\begin{aligned}
|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})+(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\
&= \mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b} \\
&= 2\mathbf{a}\cdot\mathbf{a}+2\mathbf{b}\cdot\mathbf{b} \\
&= 2|\mathbf{a}|^2+2|\mathbf{a}|^2
\end{aligned}$$

64) *Show that if* $\boldsymbol{u}+\boldsymbol{v}$ *and* $\boldsymbol{u}-\boldsymbol{v}$ *are orthogonal, then the vectors* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *must have the same length.*

Solution: If $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal, then $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=0$. Then distribute the dot product.

$$\begin{aligned}
0 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v} \\
&= |\mathbf{u}|^2-|\mathbf{v}|^2
\end{aligned}$$

Here, use the fact that the dot product is symmetric, $\mathbf{v}\cdot\mathbf{u}=\mathbf{u}\cdot\mathbf{v}$. Thus, $|\mathbf{u}|^2=|\mathbf{v}|^2$.

Section 12.4

10) *Find the vector, not with determinants, but by using properties of cross products.*

$$\mathbf{k}\times(\mathbf{i}-2\mathbf{j})$$

Solution: First, use the distributive property.

$$\mathbf{k}\times(\mathbf{i}-2\mathbf{j})=\mathbf{k}\times\mathbf{i}-2(\mathbf{k}\times\mathbf{j})$$

Then, look at $\mathbf{k}\times\mathbf{i}$ and $\mathbf{k}\times\mathbf{j}$ separately. The cross product, $\mathbf{k}\times\mathbf{i}$, must be orthogonal to $\mathbf{k}$ and $\mathbf{i}$. Hence, $\mathbf{k}\times\mathbf{i}$ must be parallel to $\mathbf{j}$. That means that there exists a constant scalar, c, such that $\mathbf{k}\times\mathbf{i}=c\mathbf{j}$.

Using the right hand rule, we can tell that $c>0$.

Finally, we know that the triple product $|\mathbf{j}\cdot(\mathbf{k}\times\mathbf{i})|$ gives the volume of the parallel pipet with sides $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. This is a unit cube with volume 1. Hence, $|\mathbf{j}\cdot(\mathbf{k}\times\mathbf{i})| = 1$. Thus,

$$1 = |\mathbf{j}\cdot(\mathbf{k}\times\mathbf{i})| = |\mathbf{j}\cdot c\mathbf{j}| = |c|$$

Since c is positive, $c = 1$

Therefore, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$. Similarly, $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$.

Finally, combine these equation.

$$\begin{aligned}\mathbf{k}\times(\mathbf{i}-2\mathbf{j}) &= \mathbf{k}\times\mathbf{i} - 2(\mathbf{k}\times\mathbf{j}) \\ &= \mathbf{j} - 2(-\mathbf{i}) \\ &= 2\mathbf{i} + \mathbf{j}\end{aligned}$$

13) State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.

(a) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

This is meaningful. The cross product, $\mathbf{b}\times\mathbf{c}$, gives a vector. Then, the dot product with $\mathbf{a}$ gives a scalar.

(b) $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$

This is NOT meaningful. The dot product, $\mathbf{b}\cdot\mathbf{c}$, gives a scalar. However, you cannot take the cross product of a vector, $\mathbf{a}$, and a scalar, $\mathbf{b}\cdot\mathbf{c}$.

(c) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

This is meaningful. The cross product, $\mathbf{b}\times\mathbf{c}$, gives a vector. Then the cross product with this vector and $\mathbf{a}$ gives another vector.

(d) $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$

This is NOT meaningful. The dot product, $\mathbf{b}\cdot\mathbf{c}$, gives a scalar. However, you cannot take the dot product of a scalar and a vector.

(e) $(\mathbf{a}\cdot\mathbf{b})\times(\mathbf{c}\cdot\mathbf{d})$

This is NOT meaningful. The dot products, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{c}\cdot\mathbf{d}$, give scalars. However, you cannot take the cross product of two scalars.

(f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

This is meaningful. The cross products, ($\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$, give vectors. Then taking the dot product of these vectors give a scalar.

20) Find two unit vectors orthogonal to both $\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i} + \boldsymbol{j}$.

These two vectors are not parallel, so the only two unit vectors that are orthogonal to both can be written

$$\frac{(\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})}{|(\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})|} \qquad \text{and} \qquad -\frac{(\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})}{|(\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})|}$$

Then, calculate

$$\begin{aligned} (\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j}) &= \mathbf{j} \times (\mathbf{i}+\mathbf{j}) - \mathbf{k} \times (\mathbf{i}+\mathbf{j}) \\ &= \mathbf{j} \times \mathbf{i} + \mathbf{j} \times \mathbf{j} - \mathbf{k} \times \mathbf{i} - \mathbf{k} \times \mathbf{j} \\ &= -\mathbf{k} + 0 - \mathbf{j} - (-\mathbf{i}) \\ &= \mathbf{i} - \mathbf{j} - \mathbf{k} \end{aligned}$$

The magnitude of this vector is

$$|\mathbf{i} - \mathbf{j} - \mathbf{k}| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}$$

Therefore, the two unit vectors we are looking for are

$$\frac{(\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})}{|(\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})|} = \frac{1}{\sqrt{3}}\big(\mathbf{i} - \mathbf{j} - \mathbf{k}\big)$$

and

$$-\frac{(\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})}{|(\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})|} = \frac{1}{\sqrt{3}}\big(-\mathbf{i} + \mathbf{j} + \mathbf{k}\big)$$

29) (a) Find a nonzero orthogonal vector to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

$$P(1,0,1), \qquad Q(-2,1,3), \qquad R(4,2,5)$$

For part (a), first find the displacement vectors $\vec{PQ}$ and $\vec{QR}$. Second, calculate $\vec{PQ} \times \vec{QR}$ to find a normal (nonzero orthogonal) vector to the plane. (Here you can also use the displacement vector $\vec{PR}$ or reverse the order of any two points.)

First, let O be the origin. Then

$$\begin{aligned}\vec{PQ} &= \vec{OQ} - \vec{OP} \\ &= \langle -2,1,3\rangle - \langle 1,0,1\rangle \\ &= \langle -3,1,2\rangle\end{aligned}$$

and

$$\begin{aligned}\vec{QR} &= \vec{OR} - \vec{OQ} \\ &= \langle 4,2,5\rangle - \langle -2,1,3\rangle \\ &= \langle 6,1,2\rangle\end{aligned}$$

Now take the cross product.

$$\vec{PQ} \times \vec{QR} = \langle -3,1,2\rangle \times \langle 6,1,2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 2 \\ 6 & 1 & 2 \end{vmatrix}$$

$$\begin{aligned} &= (2-2)\mathbf{i} - (-6-12)\mathbf{j} + (-3-6)\mathbf{k} \\ &= 0\mathbf{i} + 18\mathbf{j} - 9\mathbf{k}\end{aligned}$$

Therefore, $18\mathbf{j} - 9\mathbf{k}$ (or any nonzero multiple of this vector) is orthogonal to the plane containing P, Q and R.

For part (b), use the volume formula for a parallel pipet in the following way. Add another labeled point, S, to the plane containing P, Q and R such that $PQRS$ is a parallelogram with sides PQ, QR, RS, and SP. (Draw a picture to show yourself that this can be done with any three points that form a triangle. In this case $S(7,1,3)$, but we will not need to know the point for our calculation.) This parallelogram has exactly twice the area of the triangle, PQR. Then, form a parallel pipet by moving this parallelogram in the direction of the normal vector, $\mathbf{n}$. By orthogonality, the volume of this parallel pipet is given by the area of the parallelogram times the height of the pipet. We also know that the triple produce of three side gives the volume, so we have the following equation.

$$(\vec{PQ} \times \vec{QR}) \cdot \mathbf{n} = |\mathbf{n}| 2Area(PQR)$$

From part (a), the normal vector is $\mathbf{n} = \vec{PQ} \times \vec{QR} = 18\mathbf{j} - 9\mathbf{k}$. Thus,

$$(\vec{PQ} \times \vec{QR}) \cdot \mathbf{n} = (18\mathbf{j} - 9\mathbf{k}) \cdot (18\mathbf{j} - 9\mathbf{k}) = 81(2^2 + 1^2) = 81(5)$$

Hence,

$$|\mathbf{n}| = \sqrt{81(5)} = 9\sqrt{5}$$

Therefore, the above equation simplifies to,

$$81(5) = 2(9)\sqrt{5}Area(PQR)$$

Hence,

$$Area(PQR) = \frac{9\sqrt{5}}{2}$$

44) (a) Find all vectors $\boldsymbol{v}$ such that

$$\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, -5\rangle$$

First, notice that if we have a vector $\mathbf{v}_0$ that solves this equation, then so does the vector $\mathbf{v}_0 + \langle 1, 2, 1\rangle$. This is simply because the cross product of any vector with itself is zero. Further, we can add any multiple of $\langle 1, 2, 1\rangle$ and hence our solution will look like

$$\mathbf{v}_0 + t\langle 1, 2, 1\rangle.$$

(From your knowledge of the next section you'll recognize this as the equation for a line.)

To find such a vector, $\mathbf{v}_0$, let $\mathbf{v}_0 = \langle x, y, z\rangle$ and compute the cross product.

$$\langle 1, 2, 1\rangle \times \langle x, y, z\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ x & y & z \end{vmatrix} = \mathbf{i}(2x - y) - \mathbf{j}(z - x) + \mathbf{k}(y - 2x)$$

Setting this equal to $\langle 3, 1, -5\rangle$ gives the system of equations

$$\begin{cases} 2x - y &= 3 \\ x - z &= 1 \\ y - 2x &= -5 \end{cases}$$

Because the solution set is a line, we can arbitrarily choose one of the coordinates in order to determine the other two. Lets set $z = 0$. Then, the second equation shows that $x = 1$, and the first equation shows that $y = -3$. Therefore, the solution set is all vectors pointing to the line.

$$\langle 1, -3, 0\rangle + t\langle 3, 1, -5\rangle$$

(b) Explain why there is no vector $\boldsymbol{v}$ *such that*

$$\langle 1,2,1\rangle \times \mathbf{v} = \langle 3,1,5\rangle$$

The cross product, $\mathbf{a} \times \mathbf{b}$ must be orthogonal to $\mathbf{a}$ and $\mathbf{b}$. However, in this case

$$\langle 1,2,1\rangle \cdot \langle 3,1,5\rangle = 3+2+5 = 10 \neq 0$$

Thus, these vectors are not orthogonal, so not such vector $\mathbf{v}$ can exist.

48) If $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=0$*, show that*

$$\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$$

Take the cross product with $\mathbf{a}$ of both sides of the given equation.

$$\begin{aligned}
\mathbf{a}\times(\mathbf{a}+\mathbf{b}+\mathbf{c}) &= \mathbf{a}\times 0 \\
\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} &= 0 \\
\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{a} &= 0 \\
\mathbf{a}\times\mathbf{b} &= \mathbf{c}\times\mathbf{a}
\end{aligned}$$

The rest of the equality follows from symmetry.

53) Suppose that $\boldsymbol{a}\neq 0$.

(a) *If* $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{c}$*, does it follow that* $\boldsymbol{b}=\boldsymbol{c}$*?*

No. Counterexample $\mathbf{a}=\mathbf{i}$, $\mathbf{b}=\mathbf{j}$, and $\mathbf{c}=\mathbf{k}$. Then

$$\mathbf{a}\cdot\mathbf{b} = 0 = \mathbf{a}\cdot\mathbf{c}$$

However, $\mathbf{j}\neq\mathbf{k}$

(b) *If* $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{a}\times\boldsymbol{c}$*, does it follow that* $\boldsymbol{b}=\boldsymbol{c}$*?*

No. Counterexample solution to number 44(a) in this section.

(c) *If* $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{c}$ *and* $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{a}\times\boldsymbol{c}$*, does it follow that* $\boldsymbol{b}=\boldsymbol{c}$*?*

Yes. Assume that $\mathbf{b}-\mathbf{c}\neq 0$. By rearranging the equations we have

$$\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = 0 \qquad \text{and} \qquad \mathbf{a}\times(\mathbf{b}-\mathbf{c}) = 0$$

Hence, by the property of dot products $\mathbf{b}-\mathbf{c}$ must be orthogonal to $\mathbf{a}$. Further, by the properties of the cross product $\mathbf{b}-\mathbf{c}$ must be parallel to $\mathbf{a}$. This is a contradiction. Therefore, $\mathbf{b}-\mathbf{c}=0$, i.e. $\mathbf{b}=\mathbf{c}$.

Section 12.5

4) Find a vector equation and parametric equation for the line through the point $(0, 14, -10)$ *and parallel to the line* $x = -1 + 2t,\ y = 6 - 3t,\ z = 3 + 9t$

Solution: The slope of the line, $x = -1 + 2t,\ y = 6 - 3t,\ z = 3 + 9t$, is $\langle 2, -3, 9\rangle$. Then plugging this into the point-slope form

$$\mathbf{x} = \langle x, y, z\rangle = \langle 0, 14, -10\rangle + t\langle 2, -3, 9\rangle$$

Then separating this equation into each component gives the parametric equations

$$\begin{cases} x &= 2t \\ y &= 14 - 3t \\ z &= -10 + 9t \end{cases}$$

20) Determine whether the lines L_1 *and* L_2 *are parallel, skew, or intersecting. If they intersect, find the point of intersections.*

$$\begin{array}{llll} L_1: & x = 5 - 12t, & y = 3 + 9t, & z = 1 - 3t \\ L_2: & x = 3 + 8s, & y = -6s, & z = 7 + 2s \end{array}$$

Solution: First, look at the slope of these two lines. This can be read off of the parametric equations directly. For L_1, the slope is $\langle -12, 9, -3\rangle$. For L_2, the slope is $\langle 8, -6, 2\rangle$. These two vectors are multiples of each other.

$$-\frac{3}{2}\langle 8, -6, 2\rangle = \langle -12, 9, -3\rangle$$

Therefore, they are parallel.

To check that they are different lines, plug in $t = 0$ and $s = 0$ to find a point on each line and calculate the displacement vector. This gives $(5, 3, 1)$ in L_1 and $(3, 0, 7)$ in L_2. Then the displacement vector is

$$\langle 5 - 3, 3 - 0, 1 - 7\rangle = \langle 2, 3, -6\rangle$$

which is not a multiple of the slope. Therefore, these are distinct parallel lines.

26) Find the equation of the plane through the point $(2, 0, 1)$ *and perpendicular to the line* $x = 3t, y = 2 - t, z = 3 + 4t$.

Solution: There are many ways of doing this. Here is one way. First, let $t = 0$ to find a point on the line, $(0, 2, 3)$. Then calculate the displacement vector from this point to $(2, 0, 1)$.

$$\langle 2 - 0, 0 - 2, 1 - 3\rangle = \langle 2, -2, -2\rangle$$

This vector must be parallel to the plane we want to find. Additionally, the slope of the line, $\langle 3, -1, 4\rangle$, must also be parallel to the plane.

Now we have two vectors that must be parallel to the plane, so we can take there cross product to find the normal vector.

$$\langle 2,-2,-2\rangle \times \langle 3,-1,4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -2 \\ 3 & -1 & 4 \end{vmatrix} = \mathbf{i}(-8-2) - \mathbf{j}(8+6) + \mathbf{k}(-2+6)$$

Therefore, our normal vector is

$$\mathbf{n} = -10\mathbf{i} - 14\mathbf{j} + 4\mathbf{k}$$

Then use the equation of a plane formula, $\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$ where a is any point in the plane. Pick a is $(2, 0, 1)$. Then the plane we are looking for is

$$\begin{aligned} \langle -10, -14, 4\rangle \cdot (\mathbf{x} - \langle 2, 0, 1\rangle) &= 0 \\ -10(x-2) - 14(y-0) + 4(z-1) &= 0 \\ -10x - 14y + 4z + 16 &= 0 \end{aligned}$$

Divide the equation by 2 for simplicity.

$$-5x - 7y + 2z + 8 = 0$$

31) Fine the equation for the plane through the points $(0, 1, 1)$*,* $(1, 0, 1)$*, and* $(1, 1, 0)$*.*

Solution: The strategy will be to find two displacement vectors and take their cross product to find the normal vector to the plane. Then plug this normal vector and one of the points into the equation for a plane.

Taking the displacement from the first to the second point gives $(1, -1, 0)$. Taking the displacement from the second to the third gives $(0, 1, -1)$. Taking their cross product gives

$$\begin{aligned} \langle 1, -1, 0\rangle \times \langle 0, 1, -1\rangle &= (\mathbf{i} - \mathbf{j}) \times (\mathbf{j} - \mathbf{k}) \\ &= \mathbf{i} \times \mathbf{j} - \mathbf{j} \times \mathbf{j} - \mathbf{i} \times \mathbf{k} + \mathbf{j} \times \mathbf{k} \\ &= \mathbf{k} - 0 - (-\mathbf{j}) + \mathbf{i} \\ &= \langle 1, 1, 1\rangle \end{aligned}$$

Then, plug in $\mathbf{n} = \langle 1,1,1\rangle$ and $(0,1,1)$ into the equation for a plane.

$$\begin{aligned}
\mathbf{n}\cdot(\mathbf{x}-\mathbf{a}) &= 0 \\
\langle 1,1,1\rangle \cdot (\langle x,y,z\rangle - \langle 0,1,1\rangle) &= 0 \\
(x-0)+(y-1)+(z-1) &= 0 \\
x+y+z &= 2
\end{aligned}$$

35) Find the equation for the plane that passes through the point $(3,5,-1)$ *and contains the line* $x = 4-t, y = 2t-1, z = -3t$.

Solution: Again the strategy will be to find two vectors parallel to the plane and take their cross products to find the normal vector.

First, the slope of the line, $\langle -1,2,-3\rangle$, must be parallel to the plane. Next any displacement vector from the line to the point must be parallel to the plane. Plug in $t = 0$ to get a point on the line, $(4,-1,0)$. Then the displacement vector is

$$\langle 4-3, -1-5, 0+1\rangle = \langle 1,-6,1\rangle$$

Second, take the cross product of these two vectors to find the normal vector.

$$\langle -1,2,-3\rangle \times \langle 1,-6,1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -3 \\ 1 & -6 & 1 \end{vmatrix} = \mathbf{i}(2-18) - \mathbf{j}(-1+3) + \mathbf{k}(6-2)$$

Hence, the normal vector is $\mathbf{n} = -16\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} = \langle -16,-2,4\rangle$.

Finally, plug this into the equation for a plane.

$$\begin{aligned}
\mathbf{n}\cdot(\mathbf{x}-\mathbf{a}) &= 0 \\
\langle -16,-2,4\rangle \cdot (\langle x,y,z\rangle - \langle 3,5,-1\rangle) &= 0 \\
-16(x-3) - 2(y-5) + 4(z+1) &= 0 \\
-16x - 2y + 4z + 48 + 10 + 4 &= 0 \\
-8x - y + 2z + 31 &= 0
\end{aligned}$$

48) Where does the line through $(-3, 1, 0)$ *and* $(-1, 5, 6)$ *intersect the plane* $2x + y - z = -2$*?*

Solution: First find a parametric equation for the line passing through $(-3, 1, 0)$ and $(-1, 5, 6)$. Then plug these equations into the equation for the plane to to find when the line intersects the plane. Finally, plug this time into the parametric equation for the line to find the location of the intersection.

First, find the parametric equation. The displacement vector is $\langle 2, 4, 6\rangle$, so the vector equation for the line is

$$\mathbf{x} = \langle x, y, z\rangle = \langle -3, 1, 0\rangle + t\langle 2, 4, 6\rangle$$

Hence, the parametric equations are

$$\begin{cases} x &= -3 + 2t \\ y &= 1 + 4t \\ z &= 6t \end{cases}$$

Second, plug these equations into the equation for the plane and solve for t.

$$\begin{aligned} 2x + y - z &= -2 \\ 2(-3 + 2t) + (1 + 4t) - (6t) &= -2 \\ 4t + 4t - 6t - 6 + 1 &= -2 \\ 2t &= 3 \\ t &= \frac{3}{2} \end{aligned}$$

Finally, plug in $t = 3/2$ into the equation for the line to find the point of intersection.

$$\begin{cases} x(\frac{3}{2}) &= -3 + 2(\frac{3}{2}) &= 0 \\ y(\frac{3}{2}) &= 1 + 4(\frac{3}{2}) &= 7 \\ z(\frac{3}{2}) &= 6(\frac{3}{2}) &= 9 \end{cases}$$

Therefore, the line and plane intersect at $(0, 7, 9)$.

61) Find an equation for the plane consisting of all points that are equidistant from the points $(1, 0, -2)$ *and* $(3, 4, 0)$*.*

Solution: The only thing we know about the plane is that for every point (x, y, z) on the plane its distance to $(1, 0, -2)$ and $(3, 4, 0)$ must with equal. Using the distance formula, this can be written as

$$\begin{aligned}
\sqrt{(x-1)^2+(y-0)^2+(z+2)^2} &= \sqrt{(x-3)^2+(y-4)^2+(z-0)^2} \\
(x-1)^2+(y-0)^2+(z+2)^2 &= (x-3)^2+(y-4)^2+(z-0)^2 \\
x^2-2x+1+y^2+z^2+4z+4 &= x^2-6x+9+y^2-8y+16+z^2 \\
-2x+1+4z+4 &= -6x+9-8y+16 \\
-2x+4z+5 &= -6x-8y+25 \\
4x+8y+4z &= 20 \\
x+2y+z &= 5
\end{aligned}$$

65) Find parametric equations for the line through the point $(0, 1, 2)$ *that is parallel to the plane* $x+y+z=2$ *and perpendicular to the line* $x = 1+t, y = 1-t, z = 2t$*.*

Solution: We have a point of the line, so we only need to find the slope to right down an equation for the line. Let's look at what we know about the slope of the desired line.

First, we know that the slope needs to be parallel to the plane $x + y + z = 2$. In other words, it needs to be orthogonal to this plane's normal vector, $\langle 1, 1, 1\rangle$.

Second, we know that the slope needs to be orthogonal to the slope of the line $x = 1 + t, y = 1 - t, z = 2t$. In other words in should be orthogonal to the vector $\langle 1, -1, 2\rangle$.

Taking the cross product of these two vectors will give a vector that is orthogonal to both as desired. Hence, calculate

$$\langle 1,1,1\rangle \times \langle 1,-1,2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i}(2+1) - \mathbf{j}(2-1) + \mathbf{k}(-1-1)$$

Thus, the slope vector of the line we are trying to find is $\langle 3, -1, -2\rangle$. Finally, plug in the point we were given and this slope into the equation for a line.

$$\langle x, y, z\rangle = \langle 0, 1, 2\rangle + t\langle 3, -1, -2\rangle$$

In parametric form this is

$$\begin{cases} x &= 3t \\ y &= 1 - t \\ z &= 2 - 2z \end{cases}$$