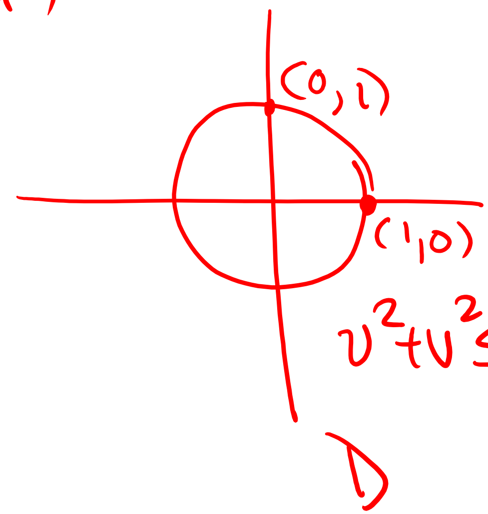
9. Consider the ellipse E given by:

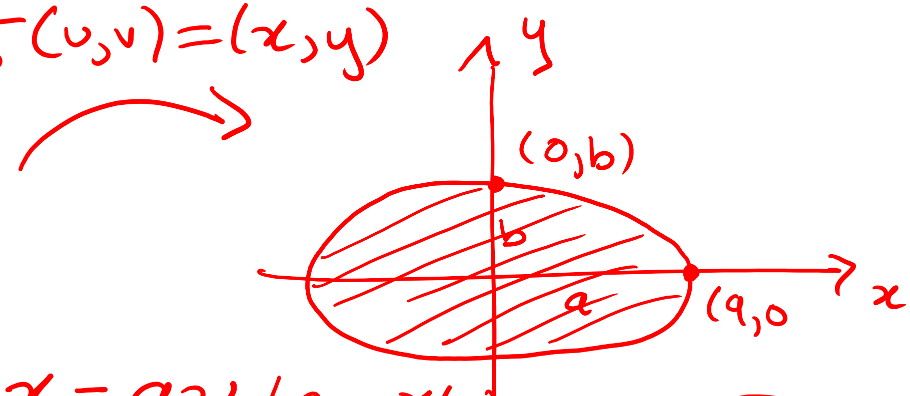
$$x^2/a^2 + y^2/b^2 = 1.$$

(a) Define a change of variables mapping the unit disk $\{u^2 + v^2 \leq 1\}$ to E. (b) Use this this to show that the area of E is πab.

(a)

$T(u,v) = (x,y)$





$x = au$, $y = bv$ $\quad\left(\begin{array}{l} u = x/a \\ v = y/b \end{array}\right)$

$E = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$

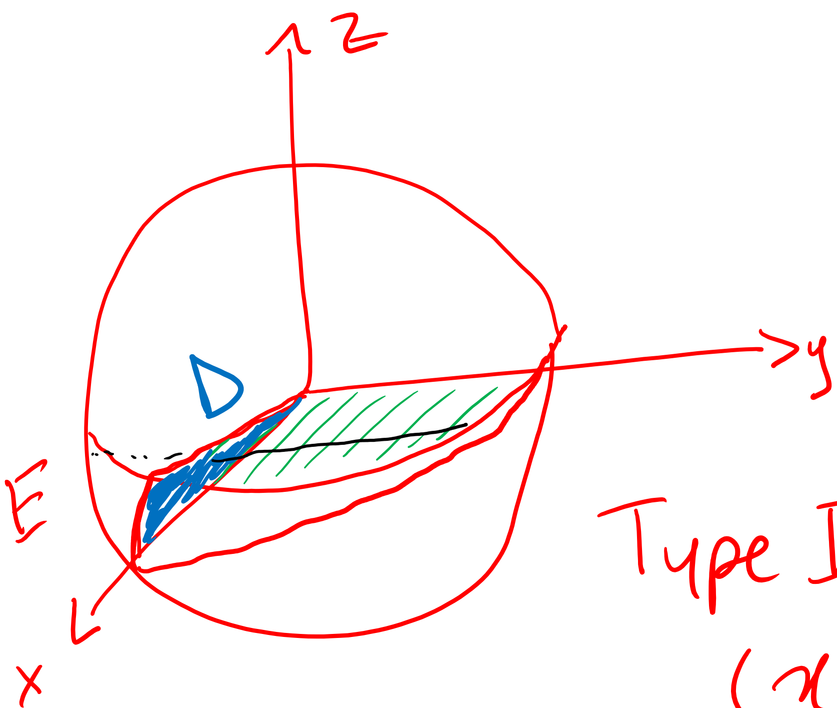
(b) $\underset{\text{of } E}{\text{Area}} = \iint_E 1\, dA = \iint_E 1\, dx\,dy = \iint_D 1 \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$

$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = \underline{\underline{ab}}$

$= \iint_D 1\, ab\, du\,dv = ab \text{ area}(D)$

$= \pi ab$ //

10. Find the volume of the region consisting of all points that are inside the sphere $x^2 + y^2 + z^2 = 4$, above the plane $z = 0$, and below the plane $z = x$.



Shadow in xz-plane ($y=0$)

$$D \begin{cases} z \geq 0 \\ x \leq z \\ x^2+z^2 \leq 4 \end{cases}$$

Type III region:

$(x,z) \in D$

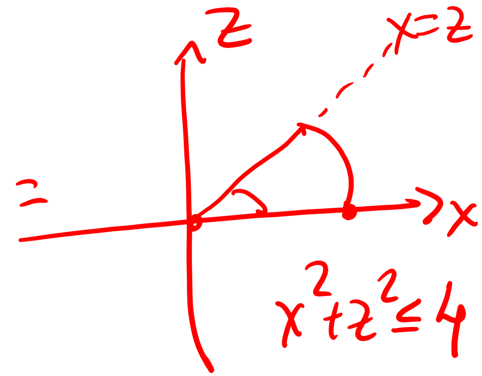
$$-\sqrt{4-x^2-z^2} \leq y \leq \sqrt{4-x^2-z^2}$$

$$\text{Vol} = \iiint_E dV = \iint_D \int_{-\sqrt{4-x^2-z^2}}^{\sqrt{4-x^2-z^2}} 1 \, dy\, dz\, dx$$

10. Find the volume of the region consisting of all points that are inside the sphere $x^2 + y^2 + z^2 = 4$, above the plane $z = 0$, and below the plane $z = x$.

$$= \iint_D 2\sqrt{4-x^2-z^2}\,dz\,dx$$

$D =$



Polar

$$= \int_0^2 \int_0^{\pi/4} \underbrace{2\sqrt{4-r^2}}\; r\,d\theta\,dr$$

$$= \begin{cases} 0 \le \theta \le \pi/4 \\ 0 \le r \le 2 \end{cases}$$

$$= \int_0^2 \frac{\pi}{4}\, 2\sqrt{4-r^2}\; r\,dr$$

$$\left(\begin{array}{l} x = r\cos\theta \\ z = r\cos\theta \end{array} \right)$$

$$= \frac{\pi}{2} \int_4^0 \left(-\frac{1}{2}\right) \sqrt{v}\; dv$$

$$\begin{array}{l} v = 4 - r^2 \\ dv = -2r\,dr \end{array}$$

$$-\frac{\pi}{4}\int_4^0 \sqrt{v}\,dv = -\frac{\pi}{4}\left.\frac{v^{3/2}}{3/2}\right|_4^0$$

$$= -\frac{\pi}{4}\frac{2}{3}\left(-4^{3/2}\right)$$

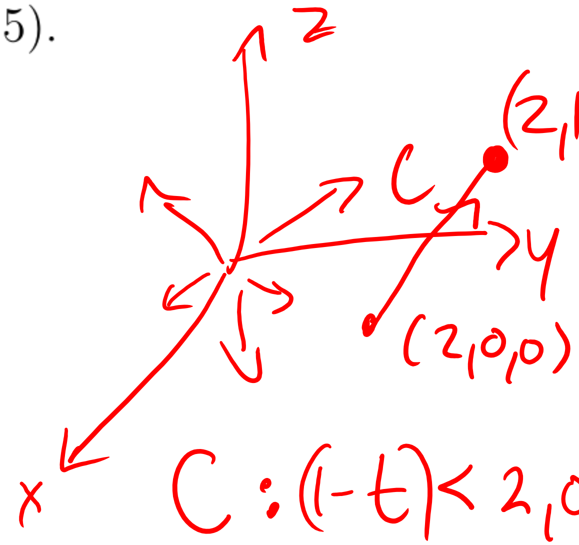
$$= \frac{\pi}{4}\frac{2}{3}2^3 \quad //$$

$$= \frac{1}{8}\underbrace{\frac{4}{3}\pi 2^3}_{\text{Vol sphere of radius 2}}$$

$$= \frac{\pi/4}{2\pi}\frac{4}{3}\pi 2^3 \quad //$$

Volume of wedge $= \pi/4 / 2\pi \, \mathrm{Vol}(\text{sphere})$

11. The force exerted by an electric charge at the origin on an electron at the point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z\rangle$ is $\mathbf{F}(\mathbf{r}) = -K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. Find the work done by this force as the electron moves along a straight line segment from $(2, 0, 0)$ to $(2, 1, 5)$.



$$\text{Work} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_0^1 \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)\,dt$$

$C: (1-t)\langle 2,0,0\rangle + t\langle 2,1,5\rangle$

$\bar{r}(t) = \langle 2, t, 5t\rangle,\ t \in [0,1]$

$$= -K\int_0^1 \frac{\bar{r}(t)}{|\bar{r}(t)|^3} \cdot \bar{r}'(t)\,dt$$

Change: $v = |\bar{r}(t)|^2$: $-\frac{K}{2}\int_{|\bar{r}(0)|^2}^{|\bar{r}(1)|^2} \frac{1}{v^{3/2}}\,dv$

$dv = 2\,\bar{r}'(t)\cdot\bar{r}(t)\,dt$

$\frac{d}{dt}\,\bar{r}(t)\cdot\bar{r}(t)$

$= \bar{r}'(t)\cdot\bar{r}(t) + \bar{r}(t)\cdot\bar{r}'(t)$

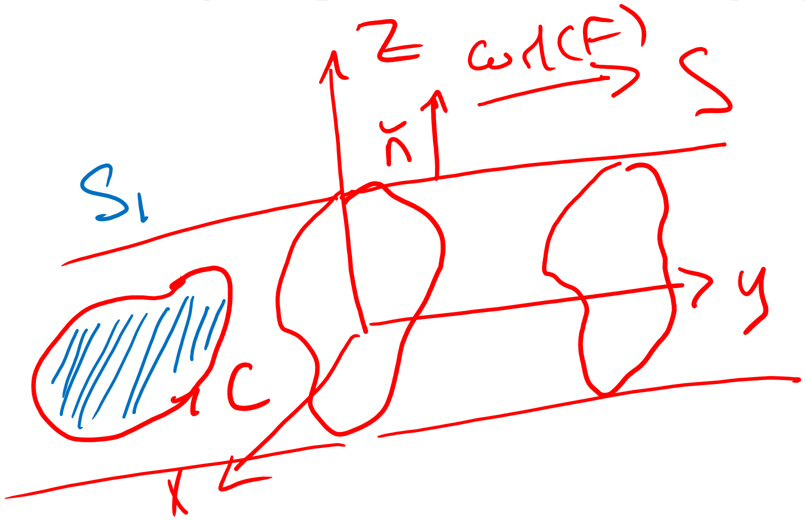
$= 2\,\bar{r}(t)\cdot\bar{r}'(t)$

$$-\frac{K}{2}\int_{|\bar{r}(0)|^2}^{|\bar{r}(1)|^2} \frac{1}{v^{3/2}}\,dv = -\frac{K}{2}\int_{4}^{30} v^{-3/2}\,dv$$

$$= -\frac{K}{2}\;\frac{v^{-1/2}}{-1/2}\Bigg|_{4}^{30} = K\left(\frac{1}{\sqrt{30}}-\frac{1}{\sqrt{4}}\right)$$

$$= K\left(\frac{1}{\sqrt{30}}-\frac{1}{2}\right)$$

12. Consider a surface S in 3−space given by an equation $z = f(x)$ (so that its trace in every plane $y = c$ is exactly the same). Show that if $\mathbf{F} = \langle x^2, y^2, xz\rangle$ then $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every simple closed curve C lying on the surface S. (hint: Stokes' theorem)



Goal: $\oint_C \bar{F}\cdot d\bar{r} = 0$

$\parallel$

Stokes: $\displaystyle\iint_{S_1} \underbrace{\mathrm{curl}(\bar{F})\cdot\bar{n}}_{\parallel}\, dS$

$\displaystyle\iint_{S_1} 0\, dS = 0$

① $\mathrm{curl}(\bar{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & xz \end{vmatrix} = \langle 0, -z, 0\rangle$

② S: level set of $g = z - f(x) = 0 \Rightarrow \bar{n} \parallel \nabla g = \langle -f_x, 0, 1\rangle$

So $\mathrm{curl}(\bar{F})\cdot\bar{n} = 0$ everywhere on S_1

13. Let $\mathbf{F} = \langle ay^2, 2y(x+z), by^2 + z^2 \rangle$. (a) For what values of a, b is $\mathbf{F}$ conservative? (b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$. (c) Using these values, give the equation of a surface S with the property that

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = 0$$

for any two points P, Q on the surface S.

(a) F is conservative $\Leftrightarrow$ curl(F) = 0 on $\mathbb{R}^3$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ ay^2 & 2y(x+z) & by^2+z^2 \end{vmatrix} = \langle 2by - 2y, 0, 2y - 2ay \rangle$$

$\equiv 0$ when

$\underline{\underline{a=1 \text{ and } b=1}}$

(b) find f: $F = \nabla f$

$$\begin{array}{ccc} \langle y^2, & 2y(x+z), & y^2+z^2 \rangle \\ \| & \| & \| \\ f_x & f_y & f_z \end{array}$$

13. Let $\mathbf{F} = \langle ay^2, 2y(x+z), by^2+z^2 \rangle$. (a) For what values of a, b is $\mathbf{F}$ conservative? (b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$. (c) Using these values, give the equation of a surface S with the property that

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = 0$$

for any two points P, Q on the surface S.

① $f_x = y^2$ ② $f_y = 2y(x+z)$ ③ $f_z = y^2 + z^2$

$f = \int y^2 dx = \underbrace{xy^2 + g(y,z)}$

$\frac{\partial}{\partial y}(xy^2 + g) = 2\cancel{x}y + g_y$

$= 2\cancel{x}y + 2yz$

by ②

$g_y = 2yz \Longrightarrow g = \int 2yz\, dy$

$= y^2 z + h(z)$

$\Longrightarrow f = xy^2 + y^2 z + h(z)$

③:

$y^2 + h'(z) = y^2 + z^2$

$\Longrightarrow h'(z) = z^2$

$\Longrightarrow h(z) = z^3/3 + \cancel{C}$

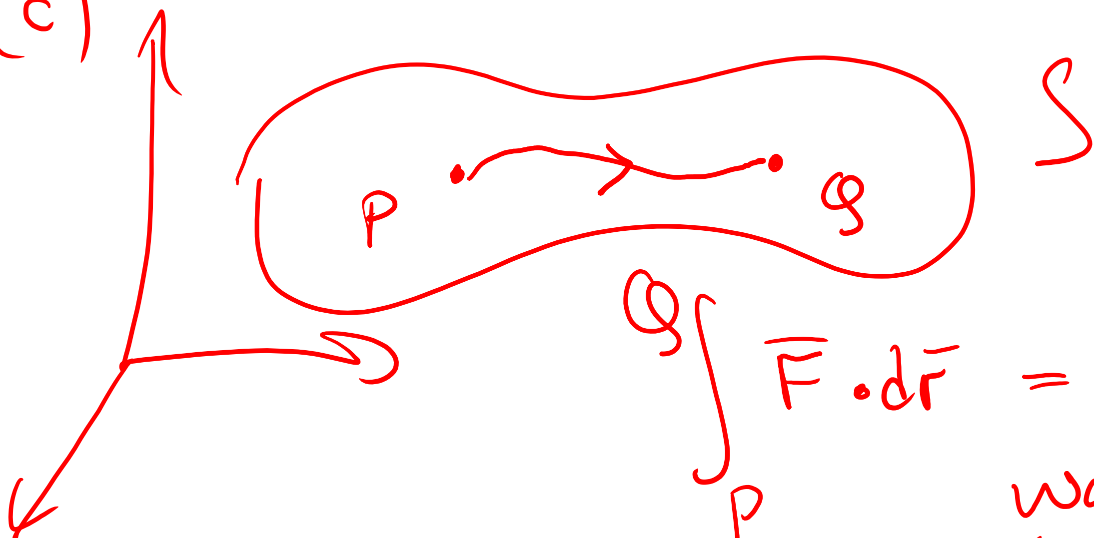
$f = xy^2 + y^2 z + z^3/3$

13. Let $\mathbf{F} = \langle ay^2, 2y(x+z), by^2 + z^2 \rangle$. (a) For what values of a, b is $\mathbf{F}$ conservative? (b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$. (c) Using these values, give the equation of a surface S with the property that

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = 0$$

for any two points P, Q on the surface S.

(c)



$\bar{F} = \nabla f$

$$\int_P^Q \bar{F} \cdot d\bar{r} = f(Q) - f(P) \underset{=}{\text{want}} 0$$

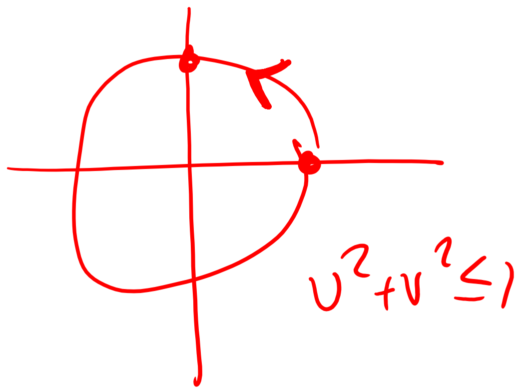
$S =$ any level surface (eg. $f = 1$).

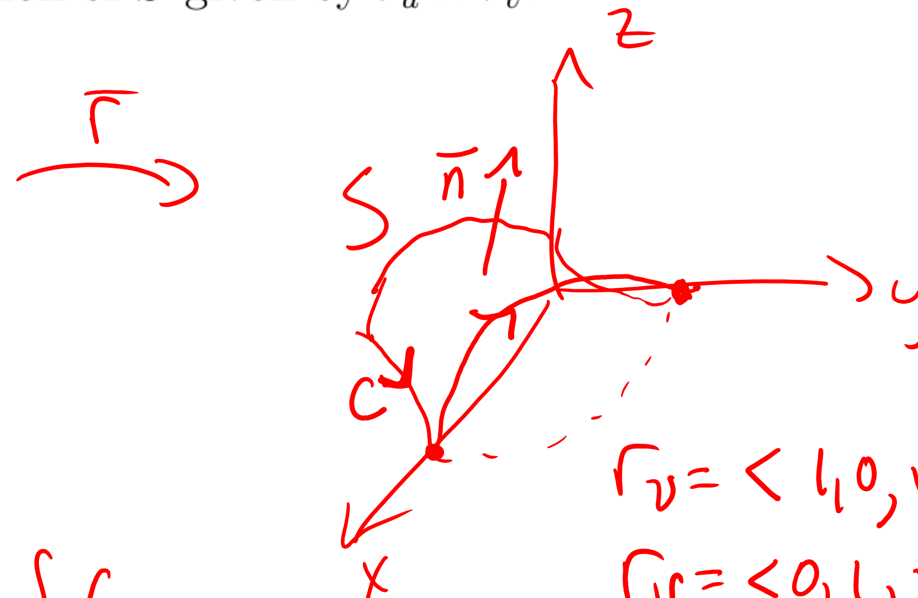
14. A surface S is parameterized by

$$r(u, v) = \langle u, v, uv \rangle \qquad u^2 + v^2 \leq 1.$$

(a) Find its surface area. (b) Parameterize the boundary curve C of S, oriented positively with respect to the orientation of S given by $r_u \times r_v$.

(a)





$r_u = \langle 1, 0, v \rangle$

$r_v = \langle 0, 1, u \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = \langle -v, -u, 1 \rangle$$

$$\text{Surface area} = \iint_S 1\, dS = \iint_D |\bar{r}_u \times \bar{r}_v|\, dA$$

$$= \iint_{u^2+v^2\leq 1} \sqrt{1+u^2+v^2}\, du\, dv \overset{\text{Polar}}{=} \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, r\, dr\, d\theta = \frac{4\pi}{3}\sqrt{2}$$

14. A surface S is parameterized by

$$r(u,v) = \langle u, v, uv \rangle \qquad u^2 + v^2 \leq 1.$$

(a) Find its surface area. (b) Parameterize the boundary curve C of S, oriented positively with respect to the orientation of S given by $r_u \times r_v$.

Boundary of D, positively oriented: $\langle \cos\theta, \sin\theta \rangle \quad \theta \in [0, 2\pi]$

$$\Big\downarrow \bar{r}$$

Boundary of S positively oriented $\langle \cos\theta, \sin\theta, \cos\theta\sin\theta \rangle \quad \theta \in [0, 2\pi]$

15. Let S be the graph of the function $f(x,y) = 2 - x^2 - y^2$ which lies above the disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$ in the xy–plane. The surface S is oriented so that the normal vector points upwards. Compute the flux

$$\int\int_S \mathbf{F} \cdot \mathbf{n} dS$$

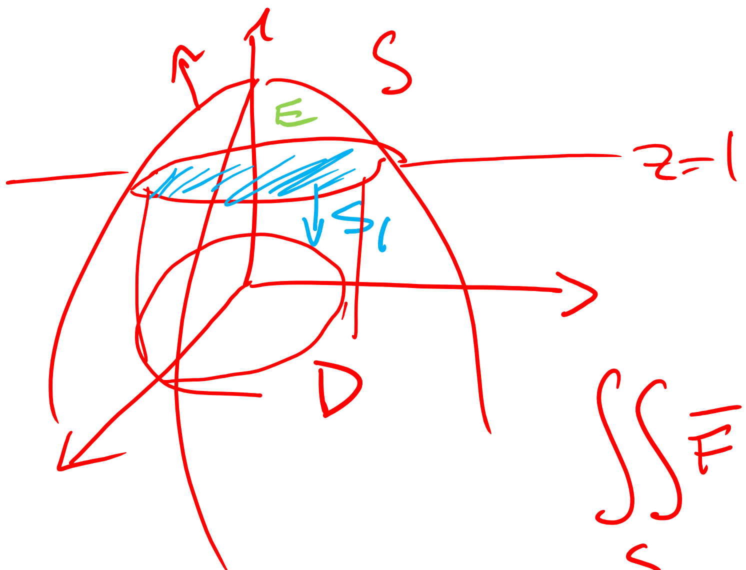
$x^2+y^2=1$

$\Rightarrow z = 2-x^2-y^2 = \underline{\underline{1}}$

of the vector field

$$\mathbf{F} = \langle -4x + \frac{x^2+y^2-1}{1+3y^2}, 3y, 7 - z - \frac{2xz}{1+3y^2} \rangle$$

through S using the divergence theorem.

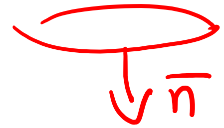


$S \cup S_1$ = bdry of

$E = \{(x,y,z):$
$(x,y) \in D$
$1 \leq z \leq 2-x^2-y^2\}$

$$\underbrace{\iint_S \vec{F}\cdot\vec{n}\,dS}_{\text{wait}} + \underbrace{\iint_{S_1} \vec{F}\cdot\vec{n}\,dS}_{\text{I}} = \underbrace{\iiint_E \operatorname{div}(F)\,dV}_{\underline{\text{II}}}$$

I] $\iint_{S_1} \bar{F}\cdot\bar{n}\,dS$



$\bar{r}(x,y) = \langle x, y, 1\rangle$

$(x,y)\in D$, $\bar{n} = \langle 0,0,-1\rangle$

$r_x = \langle 1,0,0\rangle$, $r_y = \langle 0,1,0\rangle$

$|r_x \times r_y| = 1 \Rightarrow dS = dA$

$$= \iint_D \bar{F}(\bar{r}(x,y))\cdot(-\hat{k})\,dA$$

$$= \iint_{x^2+y^2\le 1} -\left(7-1-\frac{2x}{1+y^2}\right)dx\,dy$$

$$\frac{2}{1+y^2}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,dx = 0$$

$$= -6\iint_D dx\,dy - \underbrace{\cancel{\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2x}{1+y^2}\,dx\,dy}}_{0} = \underline{\underline{-6\pi}}$$

II) $\iiint_E \operatorname{div}(F)\, dV$. $\operatorname{div}(F) = -4 + \cancel{\dfrac{2x}{1+3y^2}} + 3$

$$E \quad \boxed{\begin{array}{l} (x,y) \in D \\ 1 \leq z \leq 2 - x^2 - y^2 \end{array}}$$

$-1 - \cancel{\dfrac{2x}{1+3y^2}} = \underline{\underline{-2}}$

$$-2 \iiint_E dV = -2 \iint_D \int_1^{2-x^2-y^2} dz\, dA$$

$$= -2 \iint_D 1 - x^2 - y^2\, dA = \int_0^1 \int_0^{2\pi} (1-r^2) r\, d\theta\, dr$$

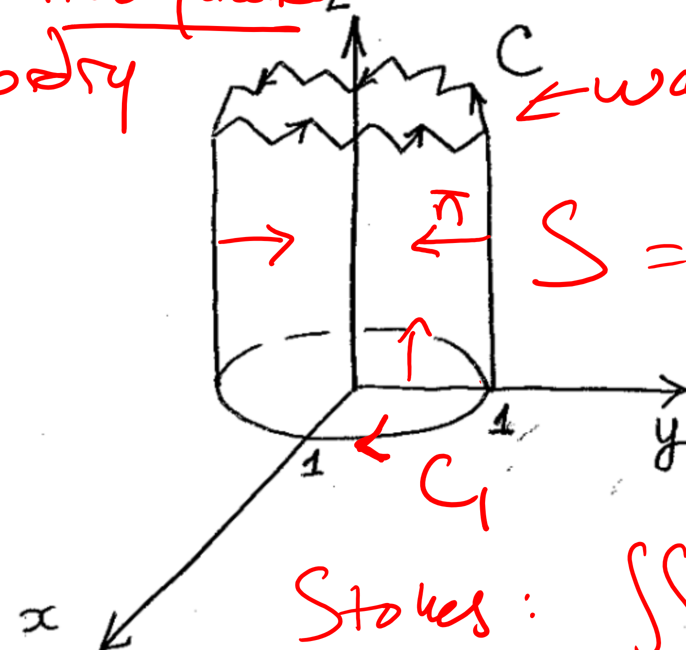
$$= 2\pi \int_0^1 r - r^3\, dr = 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4} \Big|_0^1 \right) = \underline{\underline{\pi}}$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS - 6\pi = \pi \implies \iint_S \vec{F}\cdot\vec{n}\,dS = \underline{\underline{7\pi}}.$$

16. A broken wine bottle is placed on the xy-plane as shown in the picture. It consists of a portion of a cylinder of radius 1 centered along the z–axis, and its bottom is a unit disk in the xy–plane centered at the origin. Let C be the path along the broken edge oriented as shown in the picture, and let $\mathbf{F} = \langle -y, 2x, 10z \rangle$. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

positively oriented

$C_1 \cup C$ is the bdry of S



C ← want

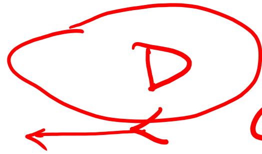
S = sides of cylinder

Stokes: $\iint_S \text{curl}(\mathbf{F}) \cdot \vec{n}\, dS = \int_C \vec{F} \cdot d\vec{r}$

$\iint_S \underbrace{\text{curl}(F) \cdot \vec{n}\, dS}^{\text{II}} = \underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{want}} + \underbrace{\int_{C_1} \vec{F} \cdot d\vec{r}}_{\text{I}}$ ← subtract

$\iint \text{curl}(F) \cdot \vec{n}\, dS = \int_{-C_1} \vec{F} \cdot d\vec{r}$

I



C_1

$r'(t) = \langle -, -, 0 \rangle$

work $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle -y, 2x, 10z \rangle$

$$= \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \qquad \vec{r}(t) \quad t \in [0, 2\pi]$$

$$= \int_{C_1} \langle -y, 2x \rangle \cdot d\vec{r} \overset{\text{Greens}}{=} -\iint_D 2+1 \, dA$$

$$= -3 \iint_D dA = \underline{\underline{-3\pi}}$$

II) $$\iint_S \operatorname{curl}(F)\cdot\bar{n}\,dS$$

$$\operatorname{curl}(F)=\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ -y & 2x & 10z \end{vmatrix} = \langle 0, 0, 2+1\rangle = \underline{\underline{\langle 0,0,3\rangle}}$$

$$\bar{n}\cdot\hat{k} = 0$$

$$\left(\begin{array}{l} g = x^2+y^2=1 \\ \nabla g = \langle 2x, 2y, 0\rangle \end{array} \right)$$

$$\operatorname{curl}(F)\cdot\bar{n} = 0+0+3\cdot 0 = \underline{\underline{0}}$$

So $$\underline{\underline{\iint_S \operatorname{curl}(F)\cdot\bar{n}\,dS = 0}}$$

$$\oint_C \vec{F} \cdot d\vec{r} - 3\pi = 0 \implies \underline{\underline{\oint_C \vec{F} \cdot d\vec{r} = 3\pi}}$$