(a) There are two unit vectors $u$ and $v$ such that the sum $u + v$ has length $1/3$.

(b) If $f(x, y)$ is continuous and both $f_x$ and $f_y$ are defined and continuous on $\mathbb{R}^2$, then $f(x, y)$ must be differentiable on $\mathbb{R}^2$.

(a) True.

\[
|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2 \langle \vec{u}, \vec{v} \rangle = 2 + 2 \cos(\theta)
\]

(b) True. Lecture 6.
(c) The work done by a vector field on a particle moving along a parameterized curve $C$ is independent of the time taken to traverse $C$, and depends only on the trajectory.

(d) The number of critical points of a differentiable function on $\mathbb{R}^2$ must be finite.

(c) \[ \mathbf{F} \rightarrow \mathbf{C} \quad \text{work} = \int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{is independent of parameterization} \]

True

(d) Critical pt. of $f(x,y)$: $f_x = f_y = 0$.

\[ f(x,y) = x^2 \]

\[ \begin{cases} f_x = 2x = 0 \\ f_y = 0 \end{cases} \]

False
(e) If \( f(x, y, z) \) is a solution of Laplace’s equation
\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0
\]
then the flux of \( \nabla f \) through the unit sphere, outwardly oriented, must be zero.

(f) If \( \mathbf{F} \) is a conservative vector field then \( \text{div}(\mathbf{F}) = 0 \).

\[
\text{div} (\nabla f) = \nabla \cdot \nabla f = 0. \quad \text{True}
\]

\[\text{Flux of } \nabla f \text{ through } S \quad \mathcal{S} \quad \text{surf. n} \quad \mathcal{S}
\]
\[
= \mathcal{SSS} \text{ div}(\nabla f) dV = 0
\]

(f) in \( \mathbb{R}^2 \): \( \mathbf{F} \) cont. \( \iff \) \( \text{curl}(\mathbf{F}) = 0 \), \( \mathbf{F} = \langle p, q \rangle \), \( p_x = p_y \).

False \( \quad \text{div}(\mathbf{F}) = 0 \iff \, p_y + p_x = 0 \).

eg. \( \mathbf{F} = \langle x, y \rangle \), \( \text{curl}(\mathbf{F}) = 0 \), \( \text{div}(\mathbf{F}) = 1 + 1 = 2 \neq 0 \).
(g) There exists a vector field \( \mathbf{F} \) such that \( \text{div}(\mathbf{F}) = x^2 + y^2 + z^2 \).

(h) There exists a vector field \( \mathbf{F} \) such that \( \text{curl}(\mathbf{F}) = \langle x^2, y^2, z^2 \rangle \).

(i) \( \mathbf{F} \) on \( \mathbb{R}^3 \). \( \text{div}(\mathbf{F}) = P_x + Q_y + R_z \)

\( \langle P, Q, R \rangle \), let \( \mathbf{F} = \langle \frac{x^3}{3}, \frac{y^2}{3}, \frac{z^3}{3} \rangle \).

\( \text{True} \)

(h) \( \mathbf{F} = \langle P, Q, R \rangle \). \( \text{Curl}(\mathbf{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \).

\( \text{Kron} : \text{div} \left( \text{curl}(\mathbf{F}) \right) = 0 \leftarrow \)

\( \text{div} \left( \langle x^2, y^2, z^2 \rangle \right) = 2x + 2y + 2z \neq 0 \)

\( \text{So there is no such } \mathbf{F} \).
(i) If \( \mathbf{F} = \langle 1/3, 1/3, 1/3 \rangle \) then the flux of \( \mathbf{F} \) across any oriented surface cannot be larger than its surface area.

(j) If the flux of \( \mathbf{F} = \langle P, Q \rangle \) across every closed curve in the plane is zero, then \( \mathbf{F} \) must be conservative.

(2) \( \mathbf{F} = \langle 1/3, 1/3, 1/3 \rangle \), \( \text{flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \leq \frac{1}{\sqrt{3}} \) True

Surface area = \( \iint_S 1 \, dS \)

So \( |\mathbf{F} \cdot \mathbf{n}| = |\mathbf{F}|/|\mathbf{n}| \cos \theta \leq \frac{1}{\sqrt{3}} \)

(iii) \( \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \leftrightarrow \text{div}(\mathbf{F}) = 0 \) eg. \( \langle y, -x \rangle \) \( \text{div} = 0 \) \( \omega d = -2 \) False
2. A particle moves along the intersection of the surfaces

\[ x^2 + y^2 + 2z^2 = 4, \quad z = xy. \]

Let \( \langle x(t), y(t), z(t) \rangle \) denote the location of the particle at time \( t \). Suppose that \( \langle x(0), y(0), z(0) \rangle = \langle 1, 1, 1 \rangle \) and \( x'(0) = 1 \). Calculate \( y'(0) \) and \( z'(0) \).

**Method 1:** Find a parameterization \( \langle x(t), y(t), z(t) \rangle \).

*Match up* \( x'(0) = 1 \), find \( y'(0), z'(0) \).

*Annoying:* \( x^2 + y^2 + 2x^2y^2 = 4 \), annoying.

**Method 2:**

\[ g(x, y, z) = x^2 + y^2 + 2z^2 = 4 \]

\[ h(x, y, z) = z - xy = 0 \]

*Goal:* couple target vector at \( \langle 1, 1, 1 \rangle \) to curve \( \langle y, 1, 1 \rangle \) lies in target plane to \( g=4 \), \( h=0 \).
Find normals to tangent planes to both surfaces:

\[ \nabla g = \langle 2y, 2y, 4z \rangle = \langle 2, 2, 4 \rangle \]

\[ \nabla h = \langle -y, -x, 1 \rangle = \langle -1, -1, 1 \rangle \]

Target vector \( \vec{v} \) is perpendicular to \( \nabla g \) and \( \nabla h \) at \( \langle 1, 1, 1 \rangle \):

So \( \vec{v} = \begin{vmatrix} i & j & k \\ 2 & 2 & 4 \\ -1 & -1 & 1 \end{vmatrix} = i (2+4) - j (2+4) + k (-2+2) = 6i - 6j \)

So \( \langle x'(0), y'(0), z'(0) \rangle = c \langle 6, -6, 0 \rangle \Rightarrow c = \frac{1}{16} \)

\( \Rightarrow y'(0) = -1, z'(0) = 0 \)
Method 3: Use total differentials \((df = f_x \, dx + f_y \, dy + f_z \, dz)\).

\[ dg = 2x \, dx + 2y \, dy + 4z \, dz = 0 \]
\[ dh = dz - y \, dx - x \, dy = 0 \]

At \( \langle 1, 1, 1 \rangle \):

\[ 2dx + 2dy + 4dz = 0 \]
\[ dz = dx + dy \]
\[ 2dx + 2dy + 4dx + 4dy = 0 \implies dy = -dx \]
\[ dz = dx - dx = 0 \]

\( x'(0) = 1, y'(0) = -1, z'(0) = 0 \).
3. Suppose $f$ is a function on $\mathbb{R}^2$ satisfying the following conditions on its directional derivatives:

$$D_{(1/\sqrt{2}, 1/\sqrt{2})} f(x, y) = \sqrt{2}x, \quad D_{(1/\sqrt{2}, -1/\sqrt{2})} f(x, y) = \sqrt{2}y.$$

(a) Find $f_x(x, y)$ and $f_y(x, y)$. (b) Assuming that $f(0, 0) = 0$, find the function $f(x, y)$.

(a) \[ D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f = \frac{1}{\sqrt{2}} f_x + \frac{1}{\sqrt{2}} f_y = \sqrt{2}x \quad \text{--- (1)} \]

\[ D_{\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f = \frac{1}{\sqrt{2}} f_x - \frac{1}{\sqrt{2}} f_y = \sqrt{2}y \quad \text{--- (2)} \]

(1) + (2): \[ \frac{2}{\sqrt{2}} f_x = \sqrt{2}(x+y) \implies f_x = x+y \]

(1) - (2): \[ \frac{2}{\sqrt{2}} f_y = \sqrt{2}(x-y) \implies f_y = x-y \]
(b) \( f_x = x+y, \ f_y = x-y. \) \textbf{Goal: find } f, \ \text{and } f(0,0) = 0.

Integrate \( f_x \) wrt \( x \): 
\[
\int f_x \, dx = \int (x+y) \, dx = \frac{x^2}{2} + xy + g(y)
\]

Plug this back into \( f_y = x-y \) :
\[
\frac{\partial}{\partial y} \left( \frac{x^2}{2} + xy + g(y) \right) = x + g'(y) = x - y
\]

\[
\Rightarrow g'(y) = -y
\]

\[
\Rightarrow g(y) = -y^2/2 + C
\]

C = 0 \text{ so }
\[
f(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2}
\]
4. Suppose that \(x, y, z\) are constrained by the equation \(g(x, y, z) = 3\). Assume that at the point \(P(0, 0, 0)\) we have \(g = 3\) and \(\nabla g = \langle 2, -1, -1 \rangle\). The equation \(g = 3\) implicitly defines \(z\) as a function of \(x\) and \(y\) in a neighborhood of the origin. Find the value of \(\frac{\partial z}{\partial x}\) at \(P\).

\[\nabla g = \langle 2, -1, -1 \rangle\]

\[g = 3\]

**Method:** Linear approximation

\[g(0, 0, 0) = 3\]

\[g(\Delta x, \Delta y, \Delta z) \approx 3 + g_x \Delta x + g_y \Delta y + g_z \Delta z\]

\[S_0 = \Delta g \approx 2 \Delta x - \Delta y - \Delta z\]

To compute \(\frac{\partial z}{\partial x}\) set \(\Delta y = 0\):

\[2 \Delta x - \Delta z \approx 0\]

\[\frac{\Delta z}{\Delta x} \approx 2\]

\[\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = 2\]
Method 2: Total differentials:

\[ g = 3 \implies 0 = dg = g_x \, dx + g_y \, dy + g_z \, dz \]

\[ = 2dx - dy - dz \]

Set \( dy = 0 \):

\[ 2dx = dz \implies \frac{\partial z}{\partial x} = 2 \]
5. (a) Find the equation of a tangent plane to the surface \( S \) given by \( 4xy - z^2 = 0 \) at \( P(1, 1, 2) \). (b) Use this to approximate the value of \( 4 \times 1.001 \times 0.99 - 2.001^2 \). (c) Find a parametric equation for the line through \( P \) perpendicular to \( S \) at \( P \).

\[ \nabla g = \langle 4y, 4x, -2z \rangle = \langle 4, 4, -4 \rangle \text{ at } P \]

is normal to \( S \).

So the tangent plane has equation:

\[ \langle 4, 4, -4 \rangle \cdot ( \vec{r} - \langle 1, 1, 2 \rangle ) = 0 \]

\[ 4x + 4y - 4z - (4 + 4 - 8) = 0 \]

\[ 4x + 4y - 4z = 0 \]
5. (a) Find the equation of a tangent plane to the surface $S$ given by $4xy - z^2 = 0$ at $P(1, 1, 2)$. (b) Use this to approximate the value of $4 \times 1.001 \times .99 - 2.001^2$. (c) Find a parametric equation for the line through $P$ perpendicular to $S$ at $P$.

\[
\mathbf{g}(1.001, 1 - .001, 2 + .001) = \frac{\Delta x}{\Delta x} = \frac{\Delta y}{\Delta y} = \frac{\Delta z}{\Delta z}
\]

\[
\approx \mathbf{g}(1,1,2) + 4\Delta x + 4\Delta y - 4\Delta z
\]

\[
= 0 + 4(.001) + 4(-.001) - 4(.001)
\]

\[
= -0.04
\]

(c) Line goes through $P(1,1,2)$ parallel to $\nabla g(1,1,2) = \langle 4, 4, -4 \rangle$

\[
\vec{r} = \langle 1, 1, 2 \rangle + t\langle 4, 4, -4 \rangle
\]
6. Use Lagrange multipliers to find the point on the surface \( g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9 \) where the function \( f(x, y, z) = 750 + 5x - 2y + 9z \) is maximized.

**Lagrange Multiplier Eqns:**

\[ \nabla g \parallel \nabla f \]

\[ g = 9 \]

\[ \nabla g = \langle 10x, 2y, 6z \rangle, \ \nabla f = \langle 5, -2, 9 \rangle. \]

At opt: \[ \langle 10x, 2y, 6z \rangle = c \langle 5, -2, 9 \rangle \Rightarrow x = \frac{5c}{10} = \frac{c}{2} \]

\[ 5 \left( \frac{c}{2} \right)^2 + c^2 + 3 \left( \frac{3c}{2} \right)^2 = 9 \]

\[ 5c^2 + 4c^2 + 27c^2 = 9 \Rightarrow 36c^2 = 36 \Rightarrow c = \pm 1 \]

\[ y = \frac{-2c}{2} = -c \]

\[ z = \frac{9c}{6} = \frac{3c}{2} \]
6. Use Lagrange multipliers to find the point on the surface \( g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9 \) where the function \( f(x, y, z) = 750 + 5x - 2y + 9z \) is maximized.

\[ 2 \text{pts} = \pm \left\langle \frac{1}{2}, -1, 3 \frac{1}{2} \right\rangle. \]

\[ f \left( \frac{1}{2}, -1, 3 \frac{1}{2} \right) = 750 + \frac{5}{2} + 2 + \frac{9 \cdot 3}{2} \]

\[ f \left( -\frac{1}{2}, -1, -3 \frac{1}{2} \right) = 750 - \frac{5}{2} - 2 - \frac{9 \cdot 3}{2} \]

So, maximum is achieved at \( \left( \frac{1}{2}, -1, 3 \frac{1}{2} \right) \).
7. Classify the critical points of the \textit{area 51} function

\[ f(x, y) = x^{51} - 51x - y^{51} + 51y \]

using the second derivative test. The reason why this function was chosen is classified.

To find critical points:

\[ f_x = 51x^{50} - 51 = 0 \]
\[ \Rightarrow x^{50} = 1 \Rightarrow x = \pm 1 \]

\[ f_y = -51y^{50} + 51 = 0 \]
\[ \Rightarrow y = \pm 1 \]

Second deriv test:

\[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} \]
\[ f_{xx} = 51 \cdot 50 \cdot x^{49} \]
\[ f_{yy} = -51 \cdot 50 \cdot y^{49} \]
\[ f_{xy} = 0 \]

\[ \begin{array}{c|c|c}
(1, 1) & <0 & \text{saddle} \\
(1, -1) & >0 & \text{local min} \\
(-1, 1) & >0 & \text{local max} \\
(-1, -1) & <0 & \text{saddle} \\
\end{array} \]
8. Evaluate by changing the order of integration:

\[ \int_0^3 \int_{x^2}^9 xe^{-y^2} \, dy \, dx. \]

Range of \( y \): \( 0 \leq y \leq 9 \)

Range of \( x \) given \( y \): \( 0 \leq x \leq \sqrt{y} \)

\[ \text{Integral} = \int_0^9 \int_0^{\sqrt{y}} xe^{-y^2} \, dx \, dy = \int_0^9 \frac{x^2}{2} e^{-y^2} \Big|_0^{\sqrt{y}} \, dy \]

\[ = \int_0^9 \frac{y}{2} e^{-y^2} \, dy \]

\[ u = y^2, \quad du = 2y \, dy \]

\[ = \frac{1}{2} \int \frac{1}{4} e^{-u} \, du = \frac{1}{4} \left[ -e^{-u} \right]_0^{81} = \frac{1-e^{-81}}{4}. \]