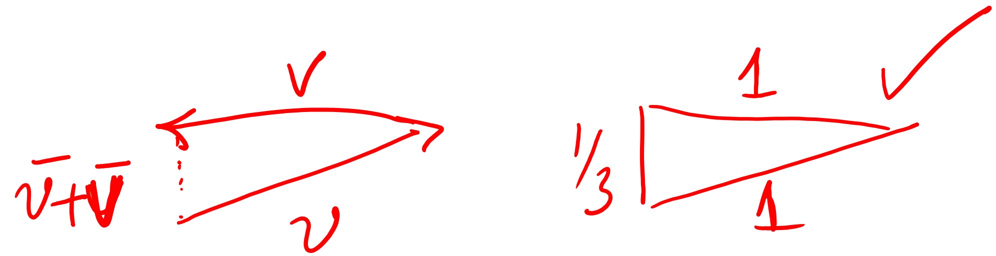
(a) There are two unit vectors u and v such that the sum $u + v$ has length $1/3$.

(b) If $f(x, y)$ is continuous and both f_x and f_y are defined and continuous on $\mathbb{R}^2$, then $f(x, y)$ must be differentiable on $\mathbb{R}^2$.

(a) True.



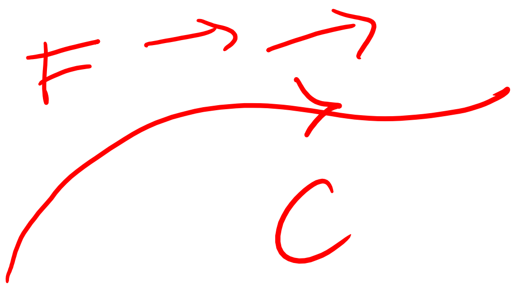
$$|\bar{v}+\bar{v}|^2 = |\bar{v}|^2 + |\bar{v}|^2 + 2\langle \bar{v}, \bar{v} \rangle$$
$$= 2 + 2\cos(\theta)$$

(b) True. Lecture 6.

(c) The work done by a vector field on a particle moving along a parameterized curve C is independent of the time taken to traverse C, and depends only on the trajectory.

(d) The number of critical points of a differentiable function on $\mathbb{R}^2$ must be finite.

(c)



work $= \int_C \vec{F} \cdot d\vec{r}$ is independent of parameterization

True

(d) Critical pt of $f(x,y)$: $f_x = f_y = 0$.

eg. $f(x,y) = x^2$. $\begin{cases} f_x = 2x = 0 \\ f_y = 0 \end{cases}$

False

(e) If $f(x, y, z)$ is a solution of Laplace's equation

$$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0 \quad \text{PDE}$$

then the flux of ∇f through the unit sphere, outwardly oriented, must be zero.

(f) If **F** is a conservative vector field then $\text{div}(F) = 0$.

(e) $\text{div}(\nabla f) = \nabla \cdot \nabla f = 0.$ True

know flux of ∇f through S $\iint_S \nabla f \cdot \vec{n}\, dS$

$= \iiint_E \text{div}(\nabla f)\, dV = 0$

(f) in $\mathbb{R}^2$: F cons $\Longleftrightarrow$ $\text{curl}(F) = 0$, $F = \langle P, Q \rangle$, $Q_x = P_y$.

False $\text{div}(F) = 0 \Longleftrightarrow Q_y + P_x = 0.$

eg. $F = \langle x, y \rangle$, $\text{curl}(F) = 0$, $\text{div}(F) = 1 + 1 = 2 \neq 0$

(g) There exists a vector field $\mathbf{F}$ such that $\text{div}(F) = x^2 + y^2 + z^2$.

(h) There exists a vector field $\mathbf{F}$ such that $\text{curl}(F) = \langle x^2, y^2, z^2 \rangle$.

(g) F on $\mathbb{R}^3$. $\text{div}(F) = P_x + Q_y + R_z$

" $\langle P, Q, R \rangle$, let $F = \langle \frac{x^3}{3}, \frac{y^3}{3}, z^3/3 \rangle$.

True

(h) $F = \langle P, Q, R \rangle$. $\text{Curl}(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix}$.

Know: $\text{div}(\text{curl}(F)) = 0 \Longleftarrow$

False

$\text{div}(\langle x^2, y^2, z^2 \rangle) = 2x + 2y + 2z \neq 0$

So there is no such F.

(i) If $\mathbf{F} = \langle 1/3, 1/3, 1/3\rangle$ then the flux of $\mathbf{F}$ across any oriented surface cannot be larger than its surface area.

(j) If the flux of $\mathbf{F} = \langle P, Q\rangle$ across every closed curve in the plane is zero, then $\mathbf{F}$ must be conservative.

(i) $F = \langle 1/3, 1/3, 1/3 \rangle$, flux $= \iint_S \underbrace{F \cdot \bar{n}}_{\leq ?\, 1} \, dS$ ✓

unit

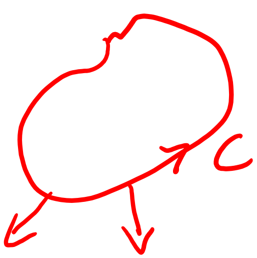
True

$|F| = \sqrt{\frac{1}{9}+\frac{1}{9}+\frac{1}{9}} = 1/\sqrt{3}$

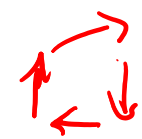
Surface area $= \iint_S 1 \, dS$

So $|F \cdot \bar{n}| = |F|\,|\bar{n}| \cos\theta \leq 1/\sqrt{3}$

(j)



flux $= \int_C F \cdot \bar{n} \, ds \Longleftrightarrow \operatorname{div}(F) = 0$ eg.



$\langle y, -x \rangle$

div=0

$\Downarrow ?$

conservative $\Longleftrightarrow \operatorname{curl}(F) = 0$

curl = −2

False

2. A particle moves along the intersection of the surfaces

$$x^2 + y^2 + 2z^2 = 4, \qquad z = xy.$$

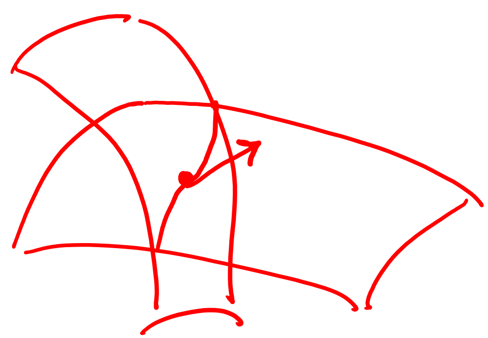
Let $\langle x(t), y(t), z(t)\rangle$ denote the location of the particle at time t. Suppose that $\langle x(0), y(0), z(0)\rangle = \langle 1, 1, 1\rangle$ and $x'(0) = 1$. Calculate $y'(0)$ and $z'(0)$.

Method1: find a parametrization $\langle x(t), y(t), z(t)\rangle$.

match up $x'(0) = 1$, find $y'(0), z'(0)$.

Annoying: $x^2+y^2+2x^2y^2=4$, annoying.

Method2:



$g(x,y,z) = x^2+y^2+2z^2=4$

$h(x,y,z) = z-xy=0$

Goal: compute tangent vector at $\langle 1,1,1\rangle$ to curve

lies in tangent planes to $g=4$, $h=0$

Find normals to tangent planes to both surfaces:

$$\nabla g = \langle 2x, 2y, 4z \rangle = \langle 2, 2, 4 \rangle$$

$$\nabla h = \langle -y, -x, 1 \rangle = \langle -1, -1, 1 \rangle$$

at $\langle 1, 1, 1 \rangle$.

tangent vector $\bar{v} \perp \nabla g$ and ∇h at $\langle 1, 1, 1 \rangle$:

So $\bar{v} = \begin{vmatrix} i & j & k \\ 2 & 2 & 4 \\ -1 & -1 & 1 \end{vmatrix} = i(2+4) - j(2+4) + k(-2+2)$

$= \underline{\underline{6\hat{i} - 6\hat{j}}}$

So $\langle \underset{\underset{1}{\parallel}}{x'(0)}, y'(0), z'(0) \rangle = c \langle 6, -6, 0 \rangle \Longrightarrow \underline{\underline{c = 1/6}}$

$\Longrightarrow y'(0) = -1, \; z'(0) = 0$ //

Method 3: Use total differentials $\left(df = f_x\,dx + f_y\,dy + f_z\,dz \right)$.

$$dg = 2x\,dx + 2y\,dy + 4z\,dz = 0$$
$$dh = dz - y\,dx - x\,dy = 0$$

describe linear approx ≡ tangent planes.

at $\langle 1,1,1\rangle$:

$$2dx + 2dy + 4dz = 0$$
$$dz = dx + dy$$
$$2dx + 2dy + 4dx + 4dy = 0 \implies \underline{\underline{dy = -dx}}$$
$$\underline{\underline{dz = dx - dx = 0}}$$

$x'(0) = 1,\quad y'(0) = -1$

$z'(0) = 0$ //

3. Suppose f is a function on $\mathbb{R}^2$ satisfying the following conditions on its directional derivatives:

$$D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(x,y) = \sqrt{2}x, \qquad D_{\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle} f(x,y) = \sqrt{2}y.$$

(a) Find $f_x(x,y)$ and $f_y(x,y)$. (b) Assuming that $f(0,0) = 0$, find the function $f(x,y)$.

(a) $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f = \frac{1}{\sqrt{2}} f_x + \frac{1}{\sqrt{2}} f_y = \sqrt{2} x$ —— ①

$D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f = \frac{1}{\sqrt{2}} f_x - \frac{1}{\sqrt{2}} f_y = \sqrt{2} y$ —— ②

① + ②: $\frac{2}{\sqrt{2}} f_x = \sqrt{2}(x+y) \Rightarrow f_x = x+y$

① − ②: $\frac{2}{\sqrt{2}} f_y = \sqrt{2}(x-y) \Rightarrow f_y = x-y$

(b) $f_x = x+y$, $f_y = x-y$. Goal: find f, $f(0,0)=0$.

Integrate f_x wrt x: $f = \int f_x \, dx = \int x+y \, dx$

$$= \frac{x^2}{2} + xy + g(y)$$

Plug this back into $f_y = x-y$: $\frac{\partial}{\partial y}\left(\frac{x^2}{2} + xy + g(y)\right)$

$$= x + g'(y) = x - y$$

$$\Rightarrow g'(y) = -y$$

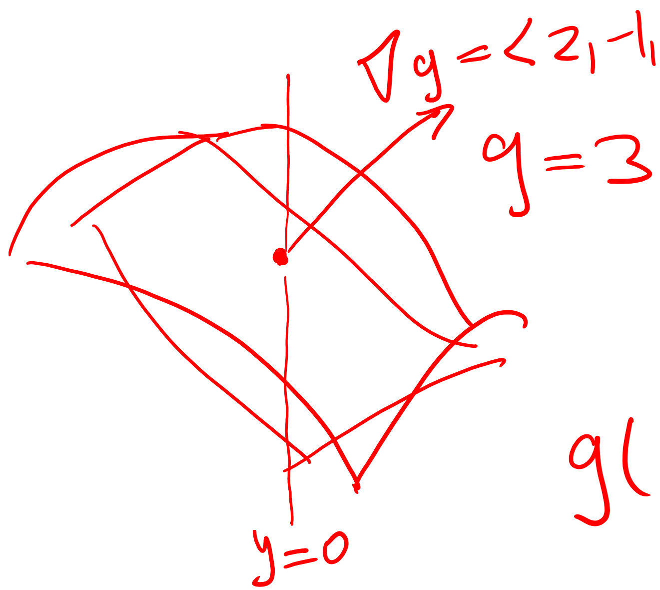
$$\Rightarrow g(y) = -y^2/2 + C$$

$$f(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2} + C$$

$C=0$ So

$$f(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2}$$

4. Suppose that x, y, z are constrained by the equation $g(x, y, z) = 3$. Assume that at the point $P(0, 0, 0)$ we have $g = 3$ and $\nabla g = \langle 2, -1, -1 \rangle$. The equation $g = 3$ implicitly defines z as a function of x and y in a neighborhood of the origin. Find the value of $\partial z/\partial x$ at P.



Method 1: linear approx.

$$g(0,0,0) = 3$$

$$g(\Delta x, \Delta y, \Delta z) \approx 3 + g_x \Delta x + g_y \Delta y + g_z \Delta z$$

$$\text{So } 0 = \Delta g \approx 2\Delta x - \Delta y - \Delta z$$

To compute $\frac{\partial z}{\partial x}$ set $\Delta y = 0$: $2\Delta x - \Delta z \approx 0$

$$\frac{\Delta z}{\Delta x} \approx 2 \xrightarrow{\text{limit}} \frac{\partial z}{\partial x} = 2$$

Method 2: Total differentials:

$$g=3: \quad 0 = dg = g_x\,dx + g_y\,dy + g_z\,dz$$

$$= 2\,dx - dy - dz$$

Set $dy=0$: $\quad 2\,dx = dz \implies \dfrac{\partial z}{\partial x} = 2$

5. (a) Find the equation of a tangent plane to the surface S given by $4xy - z^2 = 0$ at $P(1, 1, 2)$. (b) Use this to approximate the value of $4 \times 1.001 \times .99 - 2.001^2$. (c) Find a parametric equation for the line through P perpendicular to S at P.

(a) S : $g(x,y,z) = 0$ where $g(x,y,z) = 4xy - z^2$.

$\nabla g = \langle 4y, 4x, -2z \rangle = \langle 4, 4, -4 \rangle$ at P

is normal to S.

So the tangent plane has equation:

$$\boxed{\langle 4,4,-4 \rangle \cdot (\vec{r} - \langle 1,1,2 \rangle) = 0}$$

$$4x + 4y - 4z - (4 + 4 - 8) = 0$$

$$\boxed{4x + 4y - 4z = 0}$$

5. (a) Find the equation of a tangent plane to the surface S given by $4xy - z^2 = 0$ at $P(1,1,2)$. (b) Use this to approximate the value of $4 \times 1.001 \times .99 - 2.001^2$. (c) Find a parametric equation for the line through P perpendicular to S at P.

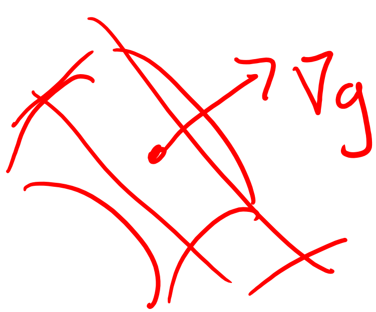
(b) $$g(\underbrace{1+.001}_{\Delta x},\ \underbrace{1-.01}_{\Delta y},\ \underbrace{2+.001}_{\Delta z})$$

$$\approx g(1,1,2) + 4\Delta x + 4\Delta y - 4\Delta z$$

$$= 0 + 4(.001) + 4(-.01) - 4(.001)$$

$$= -.04$$

(c)

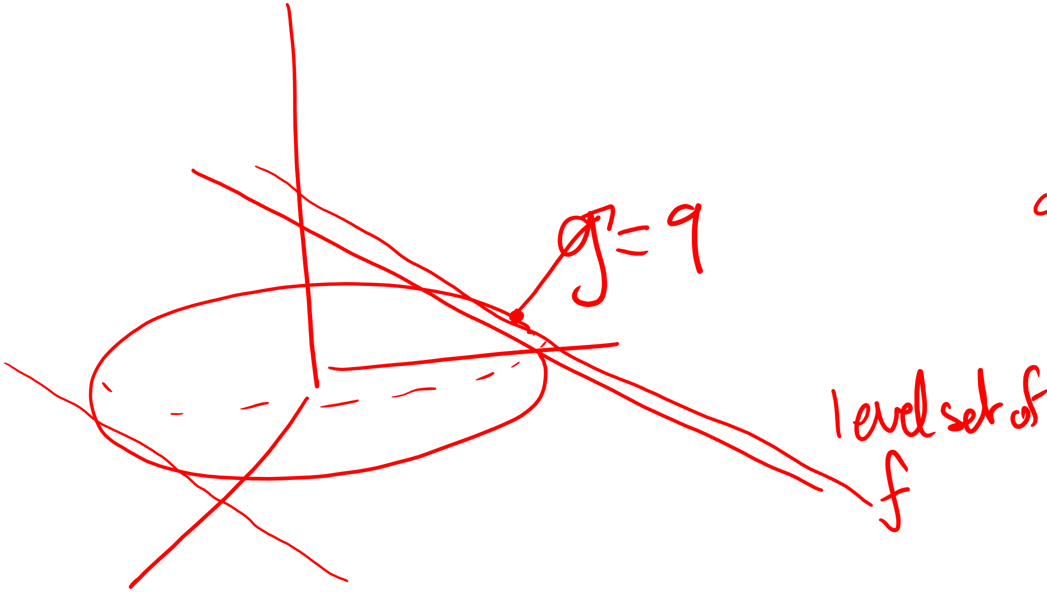


line goes through $P(1,1,2)$

parallel to $\nabla g(1,1,2) = \langle 4,4,-4 \rangle$

$$\boxed{\bar{r} = \langle 1,1,2 \rangle + t\langle 4,4,-4 \rangle}$$

6. Use Lagrange multipliers to find the point on the surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$ where the function $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximized.



Lagrange Multiplier Eqns:

at opt:

$$\begin{cases} \nabla g \parallel \nabla f \\ g = 9 \end{cases}$$

$\nabla g = \langle 10x, 2y, 6z \rangle$, $\nabla f = \langle 5, -2, 9 \rangle$.

at opt: $\langle 10x, 2y, 6z \rangle = c \langle 5, -2, 9 \rangle \Rightarrow x = \frac{5c}{10} = \frac{c}{2}$

$$\boxed{\begin{aligned} y &= \frac{-2c}{2} = -c \\ z &= \frac{9c}{6} = \frac{3c}{2} \end{aligned}}$$

$5\left(\frac{c}{2}\right)^2 + c^2 + 3\left(\frac{3c}{2}\right)^2 = 9$

$\shortparallel$

$\underbrace{5c^2 + 4c^2 + 27c^2}_{4} = 9 \Longrightarrow 36c^2 = 36, \; c = \pm 1$

6. Use Lagrange multipliers to find the point on the surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$ where the function $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximized.

2pts $= \pm \langle \frac{1}{2}, -1, \frac{3}{2} \rangle$.

maximum

$f(\frac{1}{2}, -1, \frac{3}{2}) = 750 + \frac{5}{2} + 2 + \frac{9 \cdot 3}{2}$ bigger

$f(-(\frac{1}{2}, -1, \frac{3}{2})) = 750 - \frac{5}{2} - 2 - \frac{9 \cdot 3}{2}$

So maximum is achieved at

$(\frac{1}{2}, -1, \frac{3}{2})$

7. Classify the critical points of the *area 51* function

$$f(x,y) = x^{51} - 51x - y^{51} + 51y$$

using the second derivative test. The reason why this function was chosen is classified.

To find critical pts: $f_x = 51x^{50} - 51 = 0$

$\implies x^{50} = 1 \implies x = \pm 1$

$f_y = -51y^{50} + 51 = 0$

$\implies y = \pm 1$

$x = \pm 1$ and $y = \pm 1$

Second deriv test: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$, $f_{xx} = 51 \cdot 50 x^{49}$

$f_{yy} = -51 \cdot 50 y^{49}$

$f_{xy} = 0$

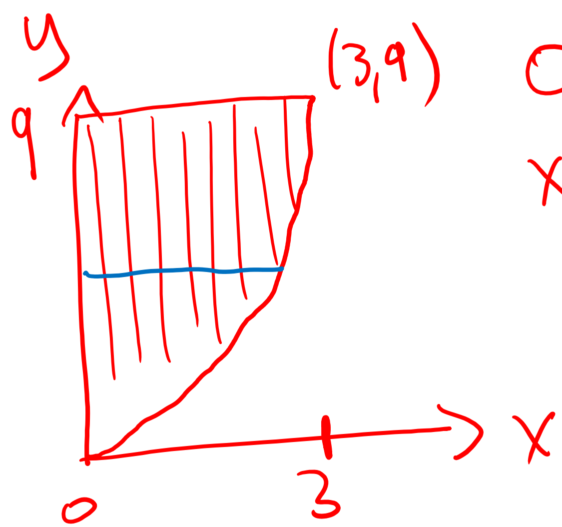
$= \begin{vmatrix} 51 \cdot 50 x^{49} & 0 \\ 0 & -51 \cdot 50 y^{49} \end{vmatrix}$

$= -(51 \cdot 50)^2 x^{49} y^{49}$

	D	type
(1, 1)	<0	saddle
(1, −1)	>0	local min
(−1, 1)	>0	local max
(−1, −1)	<0	saddle

8. Evaluate by changing the order of integration:

$$\int_0^3 \int_{x^2}^9 xe^{-y^2}\,dy\,dx.$$



$0 \le x \le 3$

$x^2 \le y \le 9$

range of y: $0 \le y \le 9$

range of x given y: $0 \le x \le \sqrt{y}$

Integral $= \displaystyle\int_0^9 \int_0^{\sqrt{y}} xe^{-y^2}\,dx\,dy = \int_0^9 \frac{x^2}{2} e^{-y^2} \Big|_0^{\sqrt{y}} dy$

$= \displaystyle\int_0^9 \frac{y}{2} e^{-y^2}\,dy$

$\boxed{u = y^2,\ du = 2y\,dy}$

$= \displaystyle\int_0^{81} \frac{1}{4} e^{-u}\,du = \frac{1}{4} \cdot -e^{-u} \Big|_0^{81} = \frac{1-e^{-81}}{4}$