Math 53 Final Exam, Prof. Srivastava
May 11, 2018, 11:40pm–2:30pm, 155 Dwinelle Hall.

Name: __________________________________________

SID: __________________________________________

GSI: __________________________________________

Name of the student to your left: ____________________________

Name of the student to your right: __________________________

Instructions: Write all answers clearly in the provided space. This exam includes some space for scratch work at the bottom of pages 2 and 6 which will not be graded. Do not under any circumstances un staple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious and will not be given full credit. You are allowed to bring one single-sided handwritten letter size cheat sheet. Calculators, phones, textbooks, and your own scratch paper are not allowed. If you are seen writing after time is up, you will lose 20 points.

When you are done, hand over your exam to your GSI unless your GSI is Shiyu Li, in which case hand it over to me.

UC Berkeley Honor Code: As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.

Sign here: __________________________________________

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Do not turn over this page until your instructor tells you to do so.
1. (12 points) Circle always true (T) or sometimes false (F) for each of the following. There is no need to provide an explanation. Two points each.

(a) Suppose $f(x, y)$ is differentiable and $f_x = 1$ and $f_y = -2$ at a point. Then there is a direction $u$ such that $D_u f = 0$ at that point. T F

(b) If the level curve of a differentiable function $g(x, y) = k$ intersects itself non-tangentially at a point $P$, then $P$ must be a critical point of $g$. T F

(c) If $a$ and $b$ are vectors in $\mathbb{R}^3$ then $a \times (a \times b)$ is always zero. T F

(d) The flux of $F = \langle x, 0, 0 \rangle$ across a sphere of radius 1 at the origin is strictly less than its flux across a sphere of radius 2 at the origin, where both are outwardly oriented. T F

(e) If curl($\nabla f$) = $\nabla f$ for a function $f$ defined on $\mathbb{R}^3$, then $f$ must be a solution of the PDE

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0. \]

T F

(f) If $S = \{(x, y, z) : f(x, y, z) = k\}$ is a level surface of a smooth function $f$ with no critical points on $S$, then $S$ must be orientable. T F

[Scratch Space Below]
2. Determine whether each of the following statements is true. If so, explain why, and if not, provide a counterexample.

(a) (3 points) If $F$ and $G$ are conservative vector fields defined on $\mathbb{R}^3$ then the sum $F + G$ is also conservative.

(b) (3 points) If $F = \langle P, Q, R \rangle$ and $G = \langle S, T, U \rangle$ are conservative vector fields defined on $\mathbb{R}^3$, then the vector field

$$H = \langle PS, QT, RU \rangle$$

with components equal to their entrywise products, is also conservative.
3. (6 points) A particle moves along the intersection of the surfaces:

\[ z = x^2 + \frac{y^2}{4} \]

and

\[ x^2 + y^2 = 25. \]

Suppose its position vector at time \( t \) is \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) and we know that \( x(0) = 3, y(0) = 4, \) and \( x'(0) = 4 \). Calculate \( y'(0) \) and \( z'(0) \).
4. Suppose \( f(x, y) = xy \) and \( x = r \cos \theta, y = r \sin \theta \).

(a) (4 points) Use the chain rule to find the partial derivatives \( \frac{\partial f}{\partial r} \) and \( \frac{\partial f}{\partial \theta} \).

(b) (4 points) Use this to approximate the value of \( f \) at the point \( (r, \theta) = (1.001, -0.01) \).
5. (6 points) Suppose $z(z(x,y))$ is a differentiable function satisfying $e^z = xyz$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial^2 z}{\partial x^2}$ as functions of $x,y,z$. 

[Scratch Space Below]
6. (6 points) Consider the function \( f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} + 5x - y \). Find and classify the critical points of the function \( g(x, y) = |\nabla f(x, y)|^2 \).
7. (6 points) Find the extreme values of $f(x, y) = e^{-xy}$ in the region $D = \{(x, y) : x^2 + 4y^2 \leq 1\}$.
8. (6 points) Compute the area of the simply connected “moustache” region enclosed by the parameterized curve

\[ \mathbf{r}(t) = \langle 5 \cos(t), \sin(t) + \cos(4t) \rangle, \quad t \in [0, 2\pi]. \]
9. (6 points) Evaluate the integral
\[ \int \int_{R} e^{x+y} \, dA \]
where \( R \) is given by the inequality \(|x| + |y| \leq 1\) by making an appropriate change of variables. (hint: sketch the region first)
10. (6 points) Find the volume of the solid that lies between the paraboloid \( z = x^2 + y^2 \) and the sphere \( x^2 + y^2 + z^2 = 2 \).
11. (6 points) Find the work done by the force field \( \mathbf{F} = (z^2, x^2, y^2) \) on a particle moving along the line segment from \((1, 0, 0)\) to \((4, 1, 2)\).
12. (6 points) A surface $S$ is parameterized by

$$\mathbf{r}(u, v) = e^{-u^2} \langle 1, \sin(v), \cos(v) \rangle,$$

where

$$0 \leq u \leq \sqrt{\pi}, \quad u^2 \leq v \leq \pi.$$

Find its surface area.
13. (6 points) Find the flux of the vector field

$$\mathbf{F}(x, y, z) = \langle y^3 z, x^3 z, 1 + e^{x^2+y^2} \rangle$$

through the paraboloid part $S$ of the boundary of the solid region

$$z + x^2 + y^2 \leq 1; \quad z \geq 0,$$

where $S$ is oriented upwards.
14. (7 points) Let $\mathbf{F}(x, y, z) = \langle yz, -xz, 1 \rangle$. Let $S$ be the portion of the paraboloid $z = 4 - x^2 - y^2$ which lies above the first octant $x \geq 0, y \geq 0, z \geq 0$; let $C$ be the closed curve $C = C_1 + C_2 + C_3$ where the curves $C_1, C_2, C_3$ are formed by intersecting $S$ with the $xy, yz,$ and $xz$ planes respectively, so that $C$ is the boundary of $S$. Orient $C$ so that it is traversed counterclockwise when seen from above in the first octant.

Use Stokes' theorem to compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by reducing it to an appropriate surface integral over $S$. 
15. (7 points) Let $S$ be the unit sphere centered at the origin, oriented outwards with normal vector $n$, and let $f(x, y, z) = x + y^2 + z^3$. Calculate

$$\int \int_S D_n f\,dS,$$

where $D_n$ is the directional derivative along $n$. 