Math 270: Interlacing Families Open Problems

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There are several (wide) open problems related to Kadison-Singer, Ramanujan graphs, and interlacing families. Here are a few (difficult) ones on which any progress is likely to be interesting.

1. **Nonbipartite Ramanujan Graphs** The result of [MSS15a] shows the existence of bipartite Ramanujan graphs of all degrees by showing that every $d-$regular graph has a signing (corresponding to a 2-lift) in which all the new eigenvalues are at most $2\sqrt{d-1}$. In particular, it does not give a lower bound on the least eigenvalue λ_n; however, because the eigenvalues of bipartite graphs are symmetric about zero, we get that $\lambda_{n-1} \geq -2\sqrt{d-1}$ for free in the bipartite case.

   It is still not known whether there are non-bipartite Ramanujan graphs of every degree. It is possible to get a bound of $4\sqrt{d-1}$ by various tricks, such as considering a double cover, but all of these amount to introducing some symmetry which allows one to control two roots for the price of one.

   So: *does every $d-$regular adjacency matrix A have a signing A_s with $\|A_s\| \leq 2\sqrt{d-1}$?* This was the original conjecture of Bilu and Linial [BL06].

2. **Higher Rank Discrepancy Conjectures** Weaver's conjecture, which we proved in class, says that given rank one symmetric matrices $A_1, \ldots, A_m$ with $\sum_{i=1}^m A_i = I$ and $\mathrm{Tr}(A_i) \leq \epsilon$, there is a partition $[m] = T_1 \cup T_2$ such that

$$\left\| \sum_{i \in T_j} A_i \right\| \leq \frac{1}{2} + O(\sqrt{\epsilon})$$

   for $j = 1, 2$.

   Is this true for positive semidefinite matrices of arbitrary rank? No nontrivial bound (better than 1) is known. Nick Harvey has shown that this is true for diagonal matrices; see [Har15] for details. The standard interlacing families approach does not work because the relevant expected characteristic polynomials are not real-rooted.

   A related discrepancy theoretic conjecture, popularized by O. Regev, is the following generalization of Spencer's six standard deviations theorem: Given arbitrary symmetric matrices $A_1, \ldots, A_m$ with $\|A_i\| \leq 1$ show that there exist signs $\epsilon_1, \ldots, \epsilon_m$ such that

$$\left\| \sum_{i=1}^{m} \epsilon_i A_i \right\| \leq O(\sqrt{n}).$$

Again, this can be solved upto a logarithmic factor using Matrix Chernoff Bounds; see the blog post [Mek14] for more details.

3. **L1 Weaver Partition / Kadison-Singer** In the language of quadratic forms, the theorem of [MSS15b] says that for any vectors $v_1, \ldots, v_m$ satisfying:

$$|\langle v_i, x\rangle|^p \leq \epsilon \cdot \sum_{i=1}^{m} |\langle v_i, x\rangle|^p \quad \forall x \in \mathbb{R}^n, i = 1, \ldots, m,$$

there is a partition $T_1 \cup T_2$ such that

$$\sum_{i \in T_j} |\langle v_i, x\rangle|^p \leq \left(\frac{1}{2} + O(\sqrt{\epsilon}\right) \cdot \sum_{i=1}^{m} |\langle v_i, x\rangle|^p \quad \forall x \in \mathbb{R}^n,$$

where $p = 2$. That is, every quadratic form of this form in which no term has too much influence can be divided into two quadratic forms which approximate it. See the blog post [Sri13] for a longer discussion of this point of view.

Is a statement like the above true for $p = 1$? A proof would imply Goddyn's Thin Tree conjecture, by taking the v_i to be incidence vectors of edges in a graph and considering zero one test vectors x. It would also imply improved embeddings of subspaces of L_1 into ℓ_1 in Banach space theory. Both are major open problems.

The issue here is quite severe, since there is no spectral theory in the $p = 1$ case. A warmup problem (which I have no idea how to solve, but which is experimentally plausible) is to show that given any line segments $L_1, \ldots, L_m$ in R^n, the Steiner polynomial

$$p(t_1, \ldots, t_m) = \mathrm{Vol}(t_1 L_1 + \ldots + t_m L_m)$$

is hyperbolic.

4. **Algorithms for Interlacing Families.** With the exception of the restricted invertibility theorem (for which the expected characteristic polynomials are simple univariate transformations of characteristic polynomials) we do not know how to efficiently compute expected characteristic polynomials or their conditionings. In fact, it is unlikely that it is possible to do so exactly in polynomial time, since for instance the matching polynomial (a very special case of the mixed characteristic polynomial) evaluated at zero counts the number of matchings in a graph, which is a $\#P$ hard problem.

Are there ways to efficiently approximate the roots of (conditional) expected characteristic polynomials, or ways to find the matrices produced by interlacing families using some other technique (such as optimization)?

5. **Probability Estimates for Interlacing Families.** We used interlacing families to prove that the norms of certain random matrices are small with nonzero probability. On the other hand, it is possible to obtain bounds that hold with high probability (via Matrix Chernoff bounds) if one is willing to lose a logarithmic factor. *Is there a way to combine the two techniques to get nontrivial probability estimates, and somehow interpolate between the two regimes?*

Note that there are examples (for instance, taking all the vectors to be standard basis vectors, as discussed in Aaron's lecture) for which the probabilities are actually exponentially small, so whatever bound you prove will probably have to include a parameter that somehow takes this into account.

References

[BL06] Yonatan Bilu and Nathan Linial. Lifts, discrepancy and nearly optimal spectral gap*. *Combinatorica*, 26(5):495–519, 2006.

[Har15] Nicholas JA Harvey. A note on the discrepancy of matrices with bounded row and column sums. *Discrete Mathematics*, 338(4):517–521, 2015.

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[MSS15a] Adam W Marcus, Daniel A Spielman, and Nikhil Srivastava. Interlacing families i: Bipartite ramanujan graphs of all degrees. *Annals of Mathematics*, 182:307–325, 2015.

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