# Lecture 9: Multivariate Barrier Arguments

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**Disclaimer**: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

## Recap

Given a graph $G = (V, E)$ we can find spectral approximators $G_1, G_2$, a two-partition of $G$ such that $G_1, G_2$ is spectrally close to $\frac{1}{2}L_G$ with $O(\sqrt{\epsilon \log n})$ error with high probability. This can be achieved using concentration bounds. Using real-stability of polynomials, we can get a $O(\sqrt{\epsilon})$ approximation using the following theorem:

**Theorem 7.1.** *If $v_1, ..., v_n$ are independent vectors such that $\sum_{i=1}^n E[v_i v_i^T] = I$ and $Tr(v_i v_i^T) = \epsilon$, then there exists some instance of $v_i$ such that $\left\| \sum_{i=1}^m v_i v_i^T \right\| < (1 + \sqrt{\epsilon})^2$. Equivalently,*

$$P\left( \left\| \sum_{i=1}^m v_i v_i^T \right\| < (1 + \sqrt{\epsilon})^2 \right) > 0$$

In order to prove this theorem, we first use the theory of interlacing families to reduce the theorem to showing $\lambda_{max}\left( E[det(xI - \sum_{i=1}^n v_i v_i^T)] \right) < (1 + \sqrt{\epsilon})^2$. Now notice that we can write the expected characteristic polynomial nicely using the following identity:

$$E[det(xI - \sum_{i=1}^n v_i v_i^T)] = \prod_i (1 - \partial_{z_i}) \, det(xI + \sum_{i=1}^n z_i E[v_i v_i^T])\Big|_{z=0}$$

Therefore, it begs the question: how does the operator $(1 - \partial_{z_i})$ change the roots of a polynomial? This will be answered using multivariate barrier functions.

## Univariate Barrier

Consider $p(x) = x^n$ and consider what the linear operator $1 - \frac{d}{dx}$ does to the roots, specifically the maximum root.

$p(x)$ has all roots at $x = 0$. Note that $p_1(x) = (1 - \frac{d}{dx})x^n$ has $n - 1$ roots at $x = 0$ and a root at $x = n$. Then, $p_2(x) = (1 - \frac{d}{dx})^2 x^n$ has $n - 2$ roots at $x = 0$ and a root at $x = n - \sqrt{n}, n + \sqrt{n}$, respectively. Note that $p_1(x)$ has a maximum root of $n$, whereas $p(x)$ has a maximum root of 0. Therefore, the maximum root was pushed a distance of $n$. Then, from $p_1(x)$ to $p_2(x)$, the maximum root was pushed a distance of only $\sqrt{n}$.

The intuition is that as $\lambda_1$ is separated more in distance from the other roots, the amount of pushing done to the maximum root becomes less. We will rigorize this notion using barrier functions.

**Definition 7.2.** The barrier function of $p(z)$ is $\phi_p(z) = (\log p)' = p'/p = \sum_{i=1}^{n} \frac{1}{z - \lambda_i}$

**Definition 7.3.** For all $\alpha < 1$, we let $\alpha_{max}(p) = \{x | \phi_p(x) \leq \alpha, x > \lambda_{max}(p)\}$.

Note that any $x \in \alpha_{max}(p)$ is a bound on $\lambda_{\max}(p)$ by definition. The idea is that when $\alpha$ is smaller, the difference between $\inf(\alpha_{max}(p))$ and $\inf(\alpha_{max}(p - p'))$ can be provably bounded.

**Theorem 7.4.** *If $x \in \alpha_{max}(p)$, then $x + 1/(1-\alpha) \in \alpha_{max}(p - p')$.*

The proof uses the following fact:

**Fact 7.5.** *$\phi_p(z)$ is convex and monotonically decreasing in the region above the roots of $p(z)$.*

*Proof.* If we let $\alpha = 1 - 1/\delta$ then it suffices to show $\phi_{p-p'}(x+\delta) \leq 1 - 1/\delta$ given $\phi_p(x) \leq 1 - 1/\delta$.

Note that $\phi_{p-p'} = \phi_{(1-\phi_p)*p} = \phi_{1-\phi_p} + \phi_p = \phi_p - \dfrac{\phi_p'}{1-\phi_p}$

Now, by convexity and monotonicity,

$$\phi_{p-p'}(x+\delta) = \phi_p(x+\delta) - \frac{\phi_p'(x+\delta)}{1-\phi_p(x+\delta)} - \phi_p(x) + \phi_p(x)$$

$$\leq \phi_p(x) + \phi_p(x+\delta) - \frac{\phi_p'(x+\delta)}{1-\phi_p(x)} - \phi_p(x)$$

$$\leq \phi_p(x) + \phi_p(x+\delta) - \delta\phi_p'(x+\delta) - \phi_p(x) \leq \phi_p(x) \leq 1 - 1/\delta$$

□

We apply this to $p(x) = x^n$. Note that $\phi_p(x) = nx^{n-1}/x^n = n/x = \alpha$. Therefore, $x = n/\alpha \in \alpha_{max}(p)$. Therefore, by repeated use of our theorem,

$$\lambda_{max}((1 - d/dx)^m p) < \frac{n}{\alpha} + \frac{m}{1-\alpha}$$

Optimizing over $\alpha \in [0, 1]$ gives $\alpha = 1/(1 + \sqrt{m/n})$ and our desired bound of

$$\lambda_{max}((1 - d/dx)^m p) < (1 + \sqrt{m/n})^2 n$$

## Multivariate Barrier

To generalize to multivariate polynomials, we say that $x$ is above $p(x)$ if $p(x + z) \neq 0$ for $z \in \mathbb{R}^n_{\geq 0}$

And we define a directional barrier function: $\phi_p^i = \partial_{z_i}(\log p(z))$.

And for $\alpha < 1$, we say $\alpha_{max}(p) = \left\{x | x \text{ is above } p, \phi_p^i(x) \leq \alpha \, \forall i\right\}$.

Then, using convexity and monotonicity arguments as in the univariate case, we get

**Theorem 7.6.** *If* $x \in \alpha_{max}(p)$*, then* $x + \dfrac{1}{1-\alpha} e_j \in \alpha_{max}(p - \partial_j p)$

Now, we want to apply this theorem to study the maximum roots of

$$E[det(xI - \sum_{i=1}^{n} v_i v_i^T)] = \prod_i (1 - \partial_{z_i})\, det(xI + \sum_{i=1}^{n} z_i E[v_i v_i^T])\Big|_{z=0} = \prod_i (1 - \partial_{y_i})\, det(\sum_{i=1}^{n} y_i E[v_i v_i^T])\Big|_{y=x}$$

Let $p(y) = det(\sum_{i=1}^{n} y_i E[v_i v_i^T]) = det(\sum_{i=1}^{n} y_i A_i)$. Note that

$$\phi_p^i = \frac{\partial_{z_i} p}{p} = \frac{det(\sum_{i=1}^{n} y_i A_i) tr((\sum_{i=1}^{n} y_i A_i)^{-1} A_i)}{det(\sum_{i=1}^{n} y_i A_i)}$$

$= tr((\sum_{i=1}^{n} y_i A_i)^{-1} A_i) = \dfrac{1}{t} tr(A_i) = \epsilon/t$ if we set $y = t\mathbf{1}$

So, we see that $\phi_p^i \leq \alpha$ when we set $\alpha = \epsilon/t$.

Since $A_i$ are positive semidefinite, note that $t\mathbf{1}$ is above $p(y) = det(\sum_{i=1}^{n} y_i A_i)$. Then, repeated use of our theorem gives:

$(t + \dfrac{1}{1-\alpha})\mathbf{1} = (t + \dfrac{1}{1-\epsilon/t})\mathbf{1}$ is above $\prod_i (1 - \partial_{z_i}) p(y)$

Therefore, the roots of $p(x) = \prod_i (1 - \partial_{y_i}) p(y)\Big|_{y=x\mathbf{1}}$ is no bigger than $t + \frac{1}{1-\epsilon/t}$ for any $t > 0$. Optimizing over $t$ gives $t = \sqrt{\epsilon} + \epsilon$ and we conclude that $\lambda_{max} < (1 + \sqrt{\epsilon})^2$