Disclaimer: These notes have not been subjected to the usual scrutiny reserved for formal publications.

Recap
Given a graph $G = (V, E)$ we can find spectral approximators $G_1, G_2$, a two-partition of $G$ such that $G_1, G_2$ is spectrally close to $\frac{1}{2}L_G$ with $O(\sqrt{\epsilon \log n})$ error with high probability. This can be achieved using concentration bounds. Using real-stability of polynomials, we can get a $O(\sqrt{\epsilon})$ approximation using the following theorem:

**Theorem 7.1.** If $v_1, ..., v_n$ are independent vectors such that $\sum_{i=1}^{n} E[v_i v_i^T] = I$ and $Tr(v_i v_i^T) = \epsilon$, then there exists some instance of $v_i$ such that $\left\| \sum_{i=1}^{m} v_i v_i^T \right\| < (1 + \sqrt{\epsilon})^2$. Equivalently,

$$P \left( \left\| \sum_{i=1}^{m} v_i v_i^T \right\| < (1 + \sqrt{\epsilon})^2 \right) > 0$$

In order to prove this theorem, we first use the theory of interlacing families to reduce the theorem to showing $\lambda_{\text{max}} \left( E[\det(xI - \sum_{i=1}^{n} v_i v_i^T)] \right) < (1 + \sqrt{\epsilon})^2$. Now notice that we can write the expected characteristic polynomial nicely using the following identity:

$$E[\det(xI - \sum_{i=1}^{n} v_i v_i^T)] = \prod_{i} (1 - \partial_z) \det(xI + \sum_{i=1}^{n} z_i E[v_i v_i^T]) \bigg|_{z=0}$$

Therefore, it begs the question: how does the operator $(1 - \partial_z)$ change the roots of a polynomial? This will be answered using multivariate barrier functions.

**Univariate Barrier**
Consider $p(x) = x^n$ and consider what the linear operator $1 - \frac{d}{dx}$ does to the roots, specifically the maximum root.

$p(x)$ has all roots at $x = 0$. Note that $p_1(x) = (1 - \frac{d}{dx})x^n$ has $n - 1$ roots at $x = 0$ and a root at $x = n$. Then, $p_2(x) = (1 - \frac{d}{dx})^2x^n$ has $n - 2$ roots at $x = 0$ and a root at $x = n - \sqrt{n}, n + \sqrt{n}$, respectively. Note that $p_1(x)$ has a maximum root of $n$, whereas $p(x)$ has a maximum root of 0. Therefore, the maximum root was pushed a distance of $n$. Then, from $p_1(x)$ to $p_2(x)$, the maximum root was pushed a distance of only $\sqrt{n}$.
The intuition is that as $\lambda_1$ is separated more in distance from the other roots, the amount of pushing done to the maximum root becomes less. We will rigorize this notion using barrier functions.

**Definition 7.2.** The barrier function of $p(z)$ is $\phi_p(z) = (\log p)' = p' / p = \sum_{i=1}^n \frac{1}{z - \lambda_i}$

**Definition 7.3.** For all $\alpha < 1$, we let $\alpha_{\text{max}}(p) = \{ x | \phi_p(x) \leq \alpha, x > \lambda_{\text{max}}(p) \}$.

Note that any $x \in \alpha_{\text{max}}(p)$ is a bound on $\lambda_{\text{max}}(p)$ by definition. The idea is that when $\alpha$ is smaller, the difference between $\inf(\alpha_{\text{max}}(p))$ and $\inf(\alpha_{\text{max}}(p - p'))$ can be provably bounded.

**Theorem 7.4.** If $x \in \alpha_{\text{max}}(p)$, then $x + 1/(1 - \alpha) \in \alpha_{\text{max}}(p - p')$.

The proof uses the following fact:

**Fact 7.5.** $\phi_p(z)$ is convex and monotonically decreasing in the region above the roots of $p(z)$.

**Proof.** If we let $\alpha = 1 - 1/\delta$ then it suffices to show $\phi_{p-p'}(x + \delta) \leq 1 - 1/\delta$ given $\phi_p(x) \leq 1 - 1/\delta$.

Note that $\phi_{p-p'} = \phi_{(1-\phi_p^p)} = \phi_1 - \phi_p = \phi_p - \frac{\phi_p^p}{1 - \phi_p}$

Now, by convexity and monotonicity,

$$\phi_{p-p'}(x + \delta) = \phi_p(x + \delta) - \frac{\phi_p^p(x + \delta)}{1 - \phi_p(x + \delta)} - \phi_p(x) + \phi_p(x)$$

$$\leq \phi_p(x) + \phi_p(x + \delta) - \frac{\phi_p^p(x + \delta)}{1 - \phi_p(x)} - \phi_p(x)$$

$$\leq \phi_p(x) + \phi_p(x + \delta) - \delta \phi_p^p(x + \delta) - \phi_p(x) \leq \phi_p(x) \leq 1 - 1/\delta$$

We apply this to $p(x) = x^n$. Note that $\phi_p(x) = n x^{n-1} / x^n = n / x = \alpha$. Therefore, $x = n / \alpha \in \alpha_{\text{max}}(p)$. Therefore, by repeated use of our theorem,

$$\lambda_{\text{max}}((1 - d/dx)^n p) < \frac{n}{\alpha} + \frac{m}{1 - \alpha}$$

Optimizing over $\alpha \in [0, 1]$ gives $\alpha = 1/(1 + \sqrt{m/n})$ and our desired bound of

$$\lambda_{\text{max}}((1 - d/dx)^n p) < (1 + \sqrt{m/n})^2 n$$

**Multivariate Barrier**

To generalize to multivariate polynomials, we say that $x$ is above $p(x)$ if $p(x + z) \neq 0$ for $z \in \mathbb{R}^n_{\geq 0}$

And we define a directional barrier function: $\phi^i_p = \partial_{z_i}(\log p(z))$.

And for $\alpha < 1$, we say $\alpha_{\text{max}}(p) = \{ x | x \text{ is above } p, \phi^i_p(x) \leq \alpha \forall i \}$.

Then, using convexity and monotonicity arguments as in the univariate case, we get
Theorem 7.6. If \( x \in \alpha_{max}(p) \), then \( x + \frac{1}{1 - \alpha} e_j \in \alpha_{max}(p - \partial_j p) \)

Now, we want to apply this theorem to study the maximum roots of

\[
E[\det(xI - \sum_{i=1}^{n} v_i v_i^T)] = \prod_{i} (1 - \partial_{z_i})\det(xI + \sum_{i=1}^{n} z_i E[v_i v_i^T])|_{z=0} = \prod_{i} (1 - \partial_{y_i})\det(\sum_{i=1}^{n} y_i E[v_i v_i^T])|_{y=x}
\]

Let \( p(y) = \det(\sum_{i=1}^{n} y_i E[v_i v_i^T]) = \det(\sum_{i=1}^{n} y_i A_i) \). Note that

\[
\phi_p^i = \frac{\partial_{z_i} p}{p} = \frac{\det(\sum_{i=1}^{n} y_i A_i)tr((\sum_{i=1}^{n} y_i A_i)^{-1})}{\det(\sum_{i=1}^{n} y_i A_i)}
\]

\[
= tr((\sum_{i=1}^{n} y_i A_i)^{-1}) = \frac{1}{t} tr(A_i) = \epsilon / t \text{ if we set } y = t1
\]

So, we see that \( \phi_p^i \leq \alpha \) when we set \( \alpha = \epsilon / t \).

Since \( A_i \) are positive semidefinite, note that \( t1 \) is above \( p(y) = \det(\sum_{i=1}^{n} y_i A_i) \). Then, repeated use of our theorem gives:

\[
(t + \frac{1}{1 - \alpha})1 = (t + \frac{1}{1 - \epsilon / t})1 \text{ is above } \prod_{i} (1 - \partial_{z_i})p(y)
\]

Therefore, the roots of \( p(x) = \prod_{i} (1 - \partial_{y_i})p(y)|_{y=x1} \) is no bigger than \( t + \frac{1}{1 - \epsilon / t} \) for any \( t > 0 \). Optimizing over \( t \) gives \( t = \sqrt{\epsilon} + \epsilon \) and we conclude that \( \lambda_{max} < (1 + \sqrt{\epsilon})^2 \).