Math 270: Geometry of Polynomials	**Fall 2015**

Lecture 3: Multiaffine Stable Polynomials

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A multiaffine polynomial is a multivariate polynomial in which each variable has degree at most one. These polynomials are quite well-understood and play a central role in the theory of stable polynomials – in particular, many properties of degree polynomials can be reduced to the multiaffine case. In this lecture we will demonstrate some of their basic properties, give a sufficient condition for a linear transformation to preserve their stability, and extend this characterization to polynomials of arbitrary degree via an operation known as *polarization*.

3.1 Multiaffine Real Stable Polynomials

We will use the notation $\mathbb{R}_1[z_1, \ldots, z_n]$ or $\mathbb{R}_{MA}[z_1, \ldots, z_n]$ to denote vector spaces of multiaffine polynomials. We begin with a characterization of multiaffine real stable polynomials. The point is that a multiaffine real polynomial is stable iff all of its bivariate restrictions are stable.

Definition 3.1 ([BBL09])**.** $f \in \mathbb{R}[z_1, \ldots, z_n]$ is *Strongly Rayleigh* if for every $i \neq j$:

$$\partial_{z_i} f(\mathbf{x}) \cdot \partial_{z_j} f(\mathbf{x}) \geq \partial_{z_i z_j} f(\mathbf{x}) \cdot f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.1}$$

Theorem 3.2 ([Brä07])**.** *A real multiaffine polynomial $f \in \mathbb{R}_1[z_1, \ldots, z_n]$ is stable iff it is Strongly Rayleigh.*

Proof. Assume f is real stable and fix $\mathbf{x} \in \mathbb{R}^n$. Consider the bivariate restriction

$$g(s, t) := f(\mathbf{x} + se_i + te_j),$$

which is a multiaffine bivariate polynomial

$$g(s, t) = a + bs + ct + dst$$

with real coefficients

$$a = f(\mathbf{x}) \quad b = \partial_{z_i} f(\mathbf{x}) \quad c = \partial_{z_j} f(\mathbf{x}) \quad d = \partial_{z_i z_j} f(\mathbf{x}).$$

Since every univariate restriction of g along a direction in the positive orthant $\mathbb{R}^2_{>0}$ is a restriction of a (real) specialization of f (by fixing all the variables other than z_i, z_j), all such restrictions are real-rooted and g is itself real stable. There are now several easy ways to show that $bc \geq ad$, and it is a good exercise to find one. For completeness: observe by applying the closure properties that for every $\lambda > 0$ the polynomial $g(\lambda r, r) = a + (\lambda b + c)r + d\lambda r^2$ must be real-rooted, whence $(\lambda b + c)^2 \geq 4ad\lambda$ for all $\lambda > 0$. If b and c are nonzero of the same sign then setting $\lambda = c/b$ yields the inequality. If they have opposite signs or if one of them is zero it is easy to see that g cannot be real stable unless it is identically zero.

For the other direction we proceed by induction. Assume $f(\mathbf{z}, z_{n+1}) = g(\mathbf{z}) + z_{n+1}h(\mathbf{z})$ is Strongly Rayleigh, with $g, h \in \mathbb{R}_1[\mathbf{z}] = \mathbb{R}_1[z_1, \dots, z_n]$. Note that both $g(\mathbf{z})$ and $h(\mathbf{z})$ are stable by the usual closure properties. Set $z_{n+1} = \alpha \in \mathbb{R}$ and observe that $g(\mathbf{z}) + \alpha h(\mathbf{z}) \in \mathbb{R}_1[\mathbf{z}]$ is Strongly Rayleigh, so by induction it must be stable for every α. If it is identically zero for some α then $g(\mathbf{z}) \equiv -\alpha h(\mathbf{z})$ and we may factor f as $f(\mathbf{z}, z_{n+1}) = (z_{n+1} - \alpha)h(\mathbf{z})$, which is stable, and we are done. Otherwise, $g(\mathbf{z}) + \alpha h(\mathbf{z}) \neq 0$ for all α and all $\mathbf{z} \in \mathcal{H}^n$, in other words

$$\Phi(\mathbf{z}) := \frac{g(\mathbf{z})}{h(\mathbf{z})} \notin \mathbb{R} \quad \forall \mathbf{z} \in \mathcal{H}^n.$$

Since Φ is continuous on $\mathcal{H}^n$ we must thus have either

$$\mathrm{Im}(\Phi(\mathbf{z})) > 0 \quad \forall \mathbf{z} \in \mathcal{H}^n \quad \textit{or}$$

$$\mathrm{Im}(\Phi(\mathbf{z})) < 0 \quad \forall \mathbf{z} \in \mathcal{H}^n.$$

In the latter case it is immediate that $f(\mathbf{z}, z_{n+1})$ is stable. In the former case, we find by changing the sign of z_{n+1} that $f(\mathbf{z}, -z_{n+1})$ must be stable. By the forward direction of the theorem, this means it is Strongly Rayleigh; applying (3.1) to the pairs $i, n+1$ we obtain the reversed inequalities:

$$\partial_{z_i} f(\mathbf{x}) \cdot \partial_{z_{n+1}} f(\mathbf{x}) \leq \partial_{z_i z_{n+1}} f(\mathbf{x}) \cdot f(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Since f is also Strongly Rayleigh these must be equalities. It is easy to check that this is only possible when $g = h$, in which case we are again done. □

Remark 3.3. The name *Strongly Rayleigh* comes from the study of electrical resistor networks via the spanning tree polynomial. The inequality (**??**) evaluated for positive vectors $\mathbf{x} > 0$ turns out to imply to "Rayleigh's Monotonicity Principle", which says that the effective resistance between two nodes in a resistive circuit cannot decrease if the resistance on any edge is increased — see [Wag05] for more details.

Remark 3.4. Multiaffine real stable polynomials satisfy a number of other striking properties. For instance, in [COSW04] it is shown that the coefficients of such a polynomial must all have the same sign and that its support (i.e., the set of multi-indices $S \subset \{0, 1\}^n$ such that x^S appears in f) must be equal to the set of bases of a matroid. Thus all multiaffine polynomials are in a sense generalizations of the spanning tree polynomial. The converse is known to be false — there are matroids which are not the support of any multiaffine real stable polynomial [Brä07].

3.2 Multiaffine Stability Preservers

In this section we derive a useful sufficient condition for establishing that a linear transformation on $\mathbb{C}_1[z_1, \dots, z_n]$ preserves stability — essentially, to show that a transformation T preserves stability of

$n-$variate polynomials, it suffices to show the stability of a single $2n-$variate generating polynomial derived from it. The proof is a good demonstration of some of the ways in which the closure properties described in the previous lecture are used.

Theorem 3.5 (Borcea-Branden [BB09a]). *Suppose $T : \mathbb{C}_1[\mathbf{z}] \to \mathbb{C}[\mathbf{z}]$ is a linear operator. If the algebraic symbol*

$$G_T(z_1, \ldots, z_n, w_1, \ldots, w_n) := T\left(\prod_{j=1}^{n}(z_j + w_j)\right) = \sum_{S \subset [n]} T(z^S) w^{[n] \setminus S}$$

is stable then T is stability-preserving.

As mentioned in the previous lecture, this condition is also necessary, but we will not prove this here; the interested reader is directed to [BB09a].

The main ingredient in the proof is the following innocuous-looking Lemma of Lieb and Sokal from 1981, which allows one to derive a stability-preserving differential operator from a stable polynomial. Note that the derivative appears in a purely formal way here, as it merely selects those terms of a polynomial which contain a certain variable.

Lemma 3.6 (Lieb-Sokal [LS81]). *Suppose $f(\mathbf{z}) + wg(\mathbf{z}) \in \mathbb{C}[\mathbf{z}, w]$ is stable and the degree of z_1 in g is at most 1. Then*

$$f(\mathbf{z}) - \partial_{z_1} g(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$$

is stable.

Proof. By the usual closure properties f and g must themselves be stable. Let $\alpha \in \mathcal{H}$ and observe that $-\alpha^{-1} \in \mathcal{H}$. Thus,

$$h(\mathbf{z}, \alpha) := \alpha g(z_1 - \alpha^{-1}, z_2, \ldots, z_n) \neq 0$$

for all $\mathbf{z} \in \mathcal{H}^n$. But since g is affine in z_1, this means that h is a multiaffine stable polynomial. Expanding in the first variable, we find that

$$h(\mathbf{z}, \alpha) = \alpha g(\mathbf{z}) - (\partial_{z_1} g)(\mathbf{z})$$

is stable, which after rearrangement means

$$\operatorname{Im}\left(\frac{-\partial_{z_1} g(\mathbf{z})}{g(\mathbf{z})}\right) \geq 0$$

for all $z \in \mathcal{H}^n$. Similarly, since $f + wg$ is stable we know that $\operatorname{Im}(f(\mathbf{z})/g(\mathbf{z})) \geq 0$ for all $\mathbf{z} \in \mathcal{H}^n$. Adding these up we have

$$\operatorname{Im}\left(\frac{-\partial_{z_1} g(\mathbf{z}) + f(\mathbf{z})}{g(\mathbf{z})}\right) \geq 0 \quad \forall \mathbf{z} \in \mathcal{H}^n,$$

so $-\partial_{z_1} g(\mathbf{z}) + f(\mathbf{z}) + vg(\mathbf{z}) \in \mathbb{R}[\mathbf{z}, v]$ is stable. Specializing v to 0 gives the result. □

Given the Lemma, Theorem 3.5 follows because every linear operator on a space of bounded degree polynomials can be written as an appropriate sum of differential operators.

Proof of Theorem 3.5. Suppose $f(\mathbf{z}) = \sum_{S\subset[n]} a_S z^S \in \mathbb{C}_1[\mathbf{z}]$ is stable. Since $w_j \mapsto -1/w_j$ preserves $\mathcal{H}$, the hypothesis implies that

$$w_1 \dots w_n G(z_1, \dots, z_n, -1/w_1, \dots - 1/w_n) = \sum_{S\subset[n]} T(z^S)(-1)^{n-|S|} w^S$$

is stable. Multiplying by $f(v_1, \dots, v_n) \in \mathbb{C}_1[v_1, \dots, v_n]$, we find that

$$\sum_{S\subset[n]} T(z^S)(-1)^{n-|S|} w^S f(v_1, \dots, v_n)$$

is stable. Since f is multiaffine, we may use the Lieb-Sokal lemma to replace each w_i by $-\partial_{v_i}$, revealing that

$$\sum_{S\subset[n]} T(z^S)(-1)^{n-|S|}(-1)^{|S|}(\partial_v^S) f(v_1, \dots, v_n)$$

is stable, where $\partial_v^S := \prod_{j\in S} \partial_{v_j}$. Setting $v_1 = \dots = v_n = 0$ preserves stability, so

$$\sum_{S\subset[n]} T(z^S)(-1)^n(\partial_v^S) f(0, \dots, 0)$$

is stable. But $(\partial_v^S f)(0, \dots, 0) = a_S$, so the above expression is equal to

$$(-1)^n \sum_{S\subset[n]} T(z^S) a_s = (-1)^n T(f).$$

Thus T preserves stability, as desired. □

3.3 Polarization and High Degree Polynomials

In this section we will introduce a tool which allows one to transfer results about multiaffine polynomials to polynomials of higher degree. The key construction is the following.

Definition 3.7 (Polarization). Given a polynomial $f \in \mathbb{C}_k[z_1, \dots, z_n]$ the *polarization* of f is the unique polynomial

$$F \in \mathbb{C}_1[z_{11}, \dots, z_{1k}, \dots, z_{n1}, \dots, z_{nk}]$$

such that

- The restriction $z_{ji} \leftarrow z_j, j = 1, \dots, n$ is equal to the original polynomial:

$$F(z_1, \dots, z_1, \dots, z_n, \dots, z_n) = f(z_1, \dots, z_n).$$

- For every $j = 1, \dots, n$, F is symmetric in $z_{j1}, \dots, z_{jn}$.

The polarization operation is denoted by $F = \Pi_k^{\uparrow}(f)$. The inverse operation is called *projection* and denoted by $f = \Pi_k^{\downarrow}(F)$.

Remark 3.8. One can also define polarization more concretely on monomials by

$$\Pi_k^{\uparrow}\left(\prod_{j=1}^{n} z_j^{\ell_j}\right) = \prod_{j=1}^{n}\left(\frac{\sum_{T\in\binom{[k]}{\ell_j}}\prod_{i\in T} z_{ji}}{\binom{k}{\ell_j}}\right),$$

which just replaces each power of z_j by an appropriate elementary symmetric polynomial.

It is trivial to see that if F is stable then $f = \Pi_k^{\downarrow}(F)$ is also stable. What is more interesting is that the converse is also true.

Theorem 3.9 ([BB09b])**.** *If $f \in \mathbb{C}_k[z_1, \ldots, z_n]$ is stable then $\Pi_k^{\uparrow}(f)$ is also stable.*

This theorem follows essentially immediately from a classical 1902 result known as the Grace-Walsh-Szego theorem. However, in order to be self-contained, we now establish it via the closure properties we have developed so far. The key new operation that we need is called *partial symmetrization*. The proof below is taken from [Pem12].

Lemma 3.10. *If $f \in \mathbb{C}_1[z_1, \ldots, z_n]$ is stable, then for every $\theta \in [0, 1]$:*

$$(1-\theta)f(z_1, z_2, \ldots, z_n) + \theta f(z_2, z_1, \ldots, z_n)$$

is stable.

Proof. By setting all the variables other than z_1, z_2 to values in $\mathcal{H}$, it is sufficient to prove the claim for bivariate polynomials. Notice that

$$T : g(z_1, z_2) \mapsto (1-\theta)g(z_1, z_2) + \theta g(z_2, z_1)$$

is a linear operator on $\mathbb{C}_1[z_1, z_2]$. Its symbol is:

$$G_T(z_1, z_2, w_1, w_2) = T((z_1+w_1)(z_2+w_2)) = z_1z_2 + w_1((1-\theta)z_2 + \theta z_1) + w_2((1-\theta)z_1 + \theta z_2) + w_1w_2.$$

We will show that G_T is Strongly Rayleigh, which by Theorem 3.2 will mean it is stable, which by Theorem 3.5 will finish the proof. By symmetry, there are only two kinds of inequalities to check:

$$\partial_{z_1}G_T \cdot \partial_{z_2}G_T - \partial_{z_1z_2}G_T \cdot G_T \geq 0,$$

and

$$\partial_{z_1}G_T \cdot \partial_{w_1}G_T - \partial_{z_1w_1}G_T \cdot G_T \geq 0,$$

where the polynomials are evaluated at real points. A simple calculation reveals that these quantities are equal to $\theta(1-\theta)(w_1 - w_2)^2$ and $\theta(z_1 - w_1)^2$ respectively, so indeed they must be nonnegative. □

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. Let $T_{ij,\theta}$ be the partial symmetrization operator which swaps indices i and j with probability θ. It can be shown by induction (excercise) that for every n there is a finite sequence of pairs $i_1 j_1, \ldots, i_N j_N$ and numbers $\theta_1, \ldots, \theta_N$ so that for every polynomial $f(z_1, \ldots, z_n)$:

$$T_{i_N j_N, \theta_N} \ldots T_{i_1 j_1, \theta_1} f = \mathbb{E}_\sigma f(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) =: \mathrm{Sym}(f),$$

where the expectation is taken over a random permutation σ of $[n]$ — i.e., we can generate a uniformly random permutation by performing appropriately biased swaps on a *predetermined* sequence of pairs. Thus, the (full) symmetrization operator $\mathrm{Sym}(f)$ preserves stability.

Let $\Pi^{\uparrow}_{k,j}$ where $j = 1, \ldots, n$ be the operator which polarizes the variable z_j only, and note that $\Pi^{\uparrow}_k = \Pi^{\uparrow}_{k,n} \circ \ldots \circ \Pi^{\uparrow}_{k,1}$. Thus it is sufficient to show that $\Pi^{\uparrow}_{k,1}$ preserves stability. By setting every $z_j, j \neq 1$ to a number in $\mathcal{H}$, it suffices to handle the univariate case. So let

$$g(z) = C \prod_{i=1}^{k} (z - \alpha_i)$$

be a univariate stable polynomial. As each $\alpha_i \notin \mathcal{H}$, each the polynomials $z_i - \alpha_i$ are stable, whence the product

$$G(z_1, \ldots, z_k) = C \prod_{i=1}^{k} (z_i - \alpha_i)$$

is stable. By the previous paragraph, $\mathrm{Sym}(G)$ must be stable; but as $\mathrm{Sym}(G)$ is symmetric in $z_1, \ldots, z_k$ and projects to g, it is equal to the polarization of g. □

Thus, a polynomial is stable iff its polarization is stable. With this fact in hand, we can easily extend Theorem 3.5 to operators acting on polynomials of bounded degree, thus obtaining the sufficiency half of Theorem 2.11 of the previous lecture, which was used there to analyze the MAP (multiaffine part) operator in the second proof of Heilmann-Lieb.

Theorem 3.11. *Suppose* $T : \mathbb{C}_k[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ *is a linear transformation and*

$$G_T(z_1, \ldots, z_n, w_1, \ldots, w_n) := T\left[(z_1 + w_1)^k \ldots (z_n + w_n)^k\right] \tag{3.2}$$

is stable. Then T *preserves stability.*

Proof. Define an operator $\Pi^{\uparrow}_k(T) : \mathbb{C}_1[z_{11}, \ldots, z_{nk}] \to \mathbb{C}_1[z_{11}, \ldots, z_{nk}]$ by

$$\Pi^{\uparrow}_k(T)(f) = \Pi^{\uparrow}_k \circ T \circ \Pi^{\downarrow}_k(f).$$

It is then easy to check that $G_{\Pi^{\uparrow}_k(T)} = \Pi^{\uparrow}_k(G_T)$. Since G_T is stable, Theorem 3.9 implies that $\Pi^{\uparrow}_k(G_T) = G_{\Pi^{\uparrow}_k(T)}$ is stable, so by Theorem 3.5 $\Pi^{\uparrow}_k(T)$ preserves stability. But $T = \Pi^{\downarrow}_k \circ \Pi^{\uparrow}_k(T) \circ \Pi^{\uparrow}_k$, which again by Theorem 3.9 is thus a composition of stability preserving operators, and must preserve stability. □

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