Math 270: Geometry of Polynomials	**Fall 2015**

Lecture 2: Stable Polynomials

Lecturer: Nikhil Srivastava *September 8*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture we develop the theory of stable polynomials, a multivariate generalization of real-rooted polynomials. These polynomials and the linear operators acting on them are remarkably well-characterized, and seemingly ad hoc results about univariate polynomials become unified and crisp when viewed as restrictions of their multivariate generalizations[1].

2.1 Stable Polynomials

We will use $\mathcal{H} = \{z : \mathrm{Im}(z) > 0\}$ to denote the open upper half plane, and $\mathbf{z}$ to denote the vector $(z_1, \ldots, z_n)$.

Definition 2.1. A nonzero polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is called *stable* if it has no zeros in $\mathcal{H}^n$, i.e.,

$$\mathrm{Im}(z_i) > 0 \quad \forall i \quad \longrightarrow \quad f(\mathbf{z}) \neq 0.$$

A stable polynomial with real coefficients is called *real stable*.

Remark 2.2. There are other notions of stability in which one requires a polynomial to be nonzero in some other region of $\mathbb{C}^n$. For instance, a Hurwitz stable polynomial is nonvanishing on the right halfplane, and Schur stable polynomial is nonvanishing on the complement of the unit disk. In this course, we will always mean upper halfplane stability unless we explicitly state otherwise.

Note that a univariate real stable polynomial must be real-rooted, since the roots of real polynomials occur in conjugate pairs. Real stability is easily seen to be equal to the following condition regarding real-rootedness of univariate restrictions. The proof is left as an exercise.

Definition 2.3. A polynomial $f(\mathbf{z})$ is real stable (resp. stable) if and only if for every $\mathbf{e} \in \mathbb{R}^n_{>0}$ and $\mathbf{x} \in \mathbb{R}^n$, the univariate restriction

$$t \mapsto f(t\mathbf{e} + \mathbf{x})$$

is real-rooted (resp. stable).

A few simple examples of stable polynomials are: the monomials $\prod_{i=1}^n z_i$, linear polynomials $\sum_{i=1} a_i z_i$ with coefficients $a_i > 0$ of the same sign, and the polynomial

$$1 - z_1 z_2, \tag{2.1}$$

[1] At some very high level this is like "kernelization" in computer science: by adding more variables it becomes possible to define a much richer class of linear transformations.

since a product of two numbers in $\mathcal{H}$ cannot be positive. Note that $1 + z_1 z_2$ is not stable.

Perhaps the most important example, which lies at the root of many applications, is the following:

Example 2.4. For positive semidefinite matrices $A_1, \ldots, A_n \succeq 0$ and Hermitian B, the determinantal polynomial

$$\det\left(\sum_{i=1}^{n} z_i A_i + B\right)$$

is real stable.

Proof. Assume that the A_i are positive definite and consider a univariate restriction

$$t \mapsto \det\left(t \sum_{i=1}^{n} e_i A_i + \left(\sum_{i=1}^{n} x_i A_i + B\right)\right).$$

Since the e_i are positive, $M := \sum_{i=1}^{n} e_i A_i \succ 0$ has a negative square root $M^{-1/2}$ and we may write the above as

$$t \mapsto \det(M^{-1/2}) \det\left(tI + M^{1/2}\left(\sum_{i=1}^{n} x_i A_i + B\right) M^{1/2}\right) \det(M^{-1/2}).$$

Since this is a multiple of a characteristic polynomial of a Hermitian matrix, it must be real-rooted.

The general positive semidefinite case is handled by taking a limit of positive definite matrices, and recalling that the limit along each univariate restriction must be real-rooted or zero. $\square$

Remark 2.5 (Helton-Vinnikov Theorem)**.** There is a stunning theorem of Lewis-Parrilo-Ramana [LPR05] (based on deep work of Helton and Vinnikov [HV07]) which says that in the bivariate case, determinantal polynomials are the *only* examples. That is, if $p(x, y)$ is real stable of degree d, then there are real symmetric $d \times d$ positive semidefinite matrices A, B and symmetric C such that

$$p(x, y) = \pm \det(xA + yB + C).$$

This is known to be false for more than 2 variables. We will get to this theorem later in the course, after introducing hyperbolic polynomials, which are a generalization of real stable polynomials.

Determinantal polynomials contain many combinatorially significant polynomials.

Example 2.6. Let $G = (V, E)$ be a connected undirected graph. Then the *spanning tree polynomial*

$$P_G(\mathbf{z}) = \sum_{\text{spanning tree } T \subset E} \prod_{e \in T} z_e$$

is real stable.

Proof. For every edge $e = (u, v) \in E$, consider the signed incidence vector $b_e = \delta_u - \delta_v$. Then the discrete Laplacian matrix of the weighted graph with edge weights z_e is given by

$$L_G(\mathbf{z}) = \sum_{e \in E} z_e b_e b_e^T.$$

It is easy to check for generic z_e this matrix has a one dimensional kernel spanned by the constant vector $\mathbf{1}$. Thus, the matrix $L_G(\mathbf{z}) + J$ has full rank for $J = \mathbf{1}\mathbf{1}^T/n$. Applying the Binet-Cauchy formula, we may write its determinant as

$$\det(L_G(\mathbf{z}) + J) = \sum_{\text{bases } B \subset \{b_e\}_{e \in E} \cup \{\mathbf{1}\}} \det\left(\sum_{v \in B} vv^T\right).$$

We encourage the reader to verify that B is a basis if and only if $B = \{b_e\}_{e \in T} \cup \{\mathbf{1}\}$ for some spanning tree T and that the determinant arising from every such B is equal to $n \cdot \prod_{e \in T} z_e$ (this is the content of the Matrix-Tree theorem). Thus,

$$\det(L_G(\mathbf{z}) + J) = n \cdot \sum_{\text{spanning tree } T \subset E} \prod_{e \in T} z_e$$

is a multiple of the spanning tree polynomial, and is real stable by example 2.4. $\square$

2.1.1 Closure Properties

Theorem 2.7. *The following linear transformations on $\mathbb{C}[\mathbf{z}]$ map every stable polynomial to another stable polynomial or to zero.*

1. **Permutation.** $f(z_1, z_2, \ldots, z_n) \mapsto f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ *for some permutation* $\sigma : [n] \to [n]$.
2. **Scaling ("External Field").** $f(z_1, \ldots, z_n) \mapsto f(az_1, \ldots, z_n)$ *where* $a > 0$.
3. **Diagonalization.** $f(z_1, z_2, \ldots, z_n) \mapsto f(z_2, z_2, z_3, \ldots, z_n) \in \mathbb{C}[z_2, \ldots, z_n]$.
4. **Inversion.** $f(z_1, \ldots, z_n) \mapsto z_1^d f(-1/z_1, \ldots, z_n)$ *where* $d = \deg_1(f)$ *is the degree of* z_1 *in* f.
5. **Specialization.** $f \mapsto f(a, z_2, \ldots, z_n) \in \mathbb{C}[z_2, \ldots, z_n]$ *where* $a \in \mathcal{H} \cup \mathbb{R}$.
6. **Differentiation.** $f \mapsto \frac{\partial}{\partial z_1} f$.

Proof. 1-3 are immediate from the definition and (4) follows because $z \mapsto -1/z$ preserves the upper half plane. (5) is clear when $a \in \mathcal{H}$ and the case $a \in \mathbb{R}$ is handled by taking a limit of the polynomials $f(a + 1/k, z_2, \ldots, z_n)$ and appealing to Definition 2.3 and Theorem **??**.

(6) is an immediate consequence of the Gauss-Lucas theorem below. $\square$

Theorem 2.8 (Gauss-Lucas). *If $f \in \mathbb{C}[z]$, the roots of $f'(z)$ lie in the convex hull of the roots of $f(z)$.*

Proof. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the roots of f and assume without loss of generality that f and f' have no common roots. If $f'(z) = 0$ then we have

$$0 = \frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{1}{z - \lambda_i} = \sum_{i=1}^{n} \frac{\overline{z - \lambda_i}}{|z - \lambda_i|^2}.$$

Rearranging and taking conjugates, we obtain

$$z = \sum_{i=1}^{n} \frac{|z - \lambda_i|^{-2}}{\sum_{i=1}^{n} |z - \lambda_i|^{-2}} \lambda_i,$$

which is a convex combination. □

Remark 2.9. The Gauss-Lucas theorem is one of the fundamental results in this area and is used very frequently. Despite its simplicity, there are several conjectured improvements of it which are still open. The most well-known of these is *Sendov's Conjecture*:

> Suppose $f(z) = (z - \lambda_1) \ldots (z - \lambda_n)$ with all λ_i in the closed unit disk $|z| \leq 1$. Then for every i, there is a root ξ of $f'(z)$ with $|\xi - \lambda_i| \leq 1$

The conjecture has been verified for polynomials of degree at most 8 [BX99]; see also [Dég14].

2.1.2 A Probabilistic Application

The elementary closure properties described above can already be used to reach interesting conclusions about random spanning tree distributions. Suppose $G = (V, E)$ is a graph, $F \subset E$ is a subset[2] of its edges, and T is a uniformly random spanning tree of G. What is the distribution of the random variable $|F \cap T|$? Again, the reason this is nontrivial is that the edges that appear in a random spanning tree are not independent.

Theorem 2.10. *The distribution of $|F \cap T|$ is a Poisson Binomial Distribution, i.e., it is the law of a sum of independent Bernoulli random variables.*

Proof. The generating polynomial of the random variable $T \cap F$ is obtained by setting all the variables $\mathbf{z}_e, e \notin F$ to 1:

$$Q_G(\mathbf{z}|_F) = P_G(\mathbf{z}_F, 1, \ldots, 1),$$

where we observe that the coefficient of the monomial $z^S := \prod_{e \in S} z_e$ in Q_G is equal to the number of spanning trees T for which $T \cap F = S$. As setting variables to real numbers preserves stability, Q_G is real stable. Thus, its diagonal restriction

$$Q_G(x, x, \ldots, x) = \sum_{k=0}^{|F|} x^k \mathbb{P}[|T \cap F| = k]$$

must be real-rooted. Normalizing by $Q_G(1, \ldots, 1)$ and applying Proposition 1.2 of Lecture 1 finishes the proof. □

[2] In applications in graph algorithms (see, e.g., the work on the travelling salesman problem [AGM$^+$10, GSS11]) one is typically interested in the case when F is a cut.

In the next two lectures, we will explore more general stability preserving operations which will allow us to obtain such conclusions for much more general distributions.

2.2 A Characterization of Stability Preserving Operators

A major reason for the recent surge of interest in the theory of real stable polynomials, and for the relative ease of its application is that there is now a complete characterization of *all* linear operators preserving stability (and real stability), and this characterization allows one to easily check which operators do so. This landmark result is due to Borcea and Branden; in this section, we state one of its simpler incarnations.

Let $\mathbb{C}_k[z_1, \ldots, z_n]$ denote the vector space of complex polynomials in $z_1, \ldots z_n$ in which each variable has degree at most k. We will call a linear transformation nondegenerate if its range has dimension at least 2.

Theorem 2.11. *A nondegenerate linear operator $T : \mathbb{C}_k[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ preserves stability iff the $2n-$variate polynomial*

$$G_T(z_1, \ldots, z_n, w_1, \ldots, w_n) := T\left[(z_1 + w_1)^k \ldots (z_n + w_n)^k\right] \tag{2.2}$$

is stable, where the operator T only acts on the z variables.

That is, the theorem says that there is a *single* $2n-$variate polynomial whose stability guarantees the stability of *all* of the $n-$variate images $T(f)$. We will present a proof of one direction of the above theorem in the case $k = 1$ in the next lecture. For now, let us display its power by using it to give an effortless alternative proof of the real-rootedness of the matching polynomial discussed in the previous lecture.

Remark 2.12. There are analogous characterization theorems for operators on $\mathbb{C}[z_1, \ldots, z_n]$ with no bound on the degree, but the corresponding symbols are power series rather than polynomials.

There are also theorems for polynomials with real coefficients, and the only difference is that the notion of nondegeneracy now requires the range to have dimension at least three.

We refer the interested reader to [BB09] for a comprehensive discussion.

2.2.1 Another Proof of Heilmann-Lieb

Given a graph G with positive edge weights $w_{uv} > 0, uv \in E$, consider the multivariate polynomial

$$Q_G(\mathbf{z}) = \prod_{uv \in E} (1 - w_{uv} z_u z_v),$$

where the variables z_v are indexed by $v \in V$. As Q_G is a product of real stable polynomials (recall (2.1)), it is real stable. Consider the *multiaffine part* operator

$$\mathrm{MAP} : \mathbb{C}[\mathbf{z}] \to \mathbb{C}_1[\mathbf{z}]$$

defined on monomials of degree at most $m := |E|$ in each variable by

$$\mathrm{MAP}\left(\prod_{e\in S} z_e^{d_e}\right) = \begin{cases}\prod_{e\in S} z_e & \text{if } d_e \le 1 \text{ for all } e\\ 0 & \text{otherwise}\end{cases}.$$

The symbol of this operator is given by

$$G_{\mathrm{MAP}}(\mathbf{z},\mathbf{w}) = \mathrm{MAP}\left(\prod_{v\in V}(z_v + w_v)^m\right) = \prod_{v\in V}(w_v^m + m z_v w_v^{m-1}) = \prod_{v\in V} w_v^{m-1}(w_v + m z_v),$$

which is real stable. Since MAP is nondegenerate, Theorem 2.11 tells us that it preserves stability. Thus,

$$\mathrm{MAP}(Q_G) = \sum_{\text{matching } M} (-1)^{|M|} \prod_{\text{edge } uv\in M} w_{uv} \prod_{\text{vertex } v\in M} z_v,$$

is real stable, and its univariate diagonal restriction $z_v \leftarrow x, v \in V$:

$$\sum_{\text{matching } M} (-1)^{|M|} x^{2|M|} \prod_{uv\in M} w_{uv}$$

is real-rooted. But this is just the reversal of $\mu_G(x)$.

2.3 Multivariate Interlacing

It seems that the two proofs of the Heilmann-Lieb theorem (using interlacing and the MAP operator) are very different, but there is in fact a close connection them.

Definition 2.13. We say that $p, q \in \mathbb{R}[z_1, \ldots, z_n]$ are in proper position, denoted $p \ll q$ if for all and $\mathbf{e} \in \mathbb{R}^n_{>0}$ and $\mathbf{x} \in \mathbb{R}^n$,

$$p(t\mathbf{e} + \mathbf{x}) \text{ interlaces } q(t\mathbf{e} + \mathbf{x}).$$

Theorem 2.14 (Borcea, Brändén [BB09][Lemma 1.8)**.** *] For any $p, q \in \mathbb{R}[z_1, \ldots, z_n]$, $p + z_{n+1}q$ is real stable if and only if $q \ll p$.*

Recalling the recurrence-based proof of Heilmann-Lieb in the first lecture, we note that the recurrence was derived from "conditioning" on one of the variables, and that the crux of the proof was to show that the two polynomials obtained by doing so interlace — precisely what is described by the theorem above.

References

[AGM+10] Arash Asadpour, Michel X Goemans, Aleksander Madry, Shayan Oveis Gharan, and Amin Saberi. An o (log n/log log n)-approximation algorithm for the asymmetric traveling salesman problem. In *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 379–389. Society for Industrial and Applied Mathematics, 2010.

[BB09] Julius Borcea and Petter Brändén. The lee-yang and pólya-schur programs. i. linear operators preserving stability. *Inventiones mathematicae*, 177(3):541–569, 2009.

[BX99] Johnny E Brown and Guangping Xiang. Proof of the sendov conjecture for polynomials of degree at most eight. *Journal of mathematical analysis and applications*, 232(2):272–292, 1999.

[Dég14] Jérôme Dégot. Sendov conjecture for high degree polynomials. *Proceedings of the American Mathematical Society*, 142(4):1337–1349, 2014.

[GSS11] Shayan Oveis Gharan, Amin Saberi, and Mohit Singh. A randomized rounding approach to the traveling salesman problem. In *Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on*, pages 550–559. IEEE, 2011.

[HV07] J William Helton and Victor Vinnikov. Linear matrix inequality representation of sets. *Communications on pure and applied mathematics*, 60(5):654–674, 2007.

[LPR05] Adrian Lewis, Pablo Parrilo, and Motakuri Ramana. The lax conjecture is true. *Proceedings of the American Mathematical Society*, 133(9):2495–2499, 2005.