Math 270: Geometry of Polynomials **Fall 2015**

Lecture 14: The Lovász Local Lemma and the Independence Polynomial

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

14.1 Lovász local lemma and the nonvanishing of the independence polynomial

An often-encountered question in probability is the following: given a collection of events $(A_x)_{x\in X}$ with $\mathbb{P}(A_x) = p_x$, when can we say that $\mathbb{P}(\cap_{x\in X}\bar{A}_x) > 0$, i.e. none of the events A_x occur with positive probability? If the events in question A_x are independent, then this is trivial to answer:

$$\mathbb{P}(\bigcap_{x\in X}\bar{A}_x) = \prod_{x\in X}(1-p_x) > 0,$$

iff $p_x \in [0,1)$ for all x. When the events are not independent, an answer is given by the Lovász local lemma (and its variations).

Definition 14.1. We say that G is the dependency graph of $(A_x)_{x\in X}$ if for all $x \in X$, A_x is independent of the σ-algebra generated by the collection $\{A_y \mid y \notin \Gamma^*(x)\}$ (where $\Gamma(x)$ is the set of neighbours of x in G and $\Gamma^*(x) := \Gamma(x) \cup \{x\}$).

Theorem 14.2 (Lovász local lemma)**.** *Let G be the dependency graph for the of events $(A_x)_{x\in X}$, and suppose that $(r_x)_{x\in X} \in [0,1)^X$ such that, for each x,*

$$\mathbb{P}(A_x) \le r_x \prod_{y\in\Gamma(x)}(1-r_y).$$

Then

$$P(\bigcap_{x\in X}\bar{A}_x) \ge \prod_{x\in X}(1-r_x) > 0.$$

Definition 14.3. $\mathbf{p} = \{p_x\}_{x\in X}$ is said to be good for G if for any family of events $(A_x)_{x\in X}$ with dependency graph G, and $\mathbb{P}(A_x) \le p_x$, we have $\mathbb{P}(\cap_{x\in X}\bar{A}_x) > 0$.

Example 14.4. Lovász local lemma states that $\mathbf{p}$ defined by $p_x = r_x\prod_{y\in\Gamma(y)}(1-r_y)$ is good for G. This, in particular, implies that if G has maximum degree Δ, then $\mathbf{p}$ with $p_x = \frac{1}{e\Delta}$ is good for G (see Remark 14.10).

In this note we will see that the conclusion of Lovász local lemma holds for dependency graph G and probabilities $(p_x)_{x\in X}$ iff the independent-set polynomial of G is nonvanishing a polydisc of radii $(p_x)_{x\in X}$.

Definition 14.5. The (multivariate) independence polynomial of G is

$$Z_G(\mathbf{w}) := \sum_{\substack{X' \subset X \\ X' \text{ independent}}} \prod_{x \in X'} w_x.$$

Since the empty set is vacuously independent, we have $Z_G(\mathbf{0}) = 1$. Also note that deg($Z_G(\mathbf{w})$)= the size of a maximal independent set of G.

Theorem 14.6. *$\mathbf{p}$ is good for G iff Z_G is nonvanishing in the closed polydisc $\bar{D}_{\mathbf{p}} := \{\mathbf{w} \in \mathbb{C}^X \mid |w_x| \leq p_x, \forall x \in X\}$.*

To prepare for the proof of this result let us begin with some notations. For all $S \subseteq X$, let

$$Z_S(\mathbf{w}) := Z_G(\mathbf{w}\mathbf{1}_S) = \sum_{\substack{X' \subseteq S \\ X' \text{ independent}}} \prod_{x \in X'} w_x,$$

where $(\mathbf{w}\mathbf{1}_S)_x = w_x \mathbf{1}_{x \in S}$. Also let

$$Z_G(\mathbf{w}; S) := \sum_{\substack{S \subseteq X' \\ X' \text{ independent}}} \prod_{x \in X'} w_x.$$

We will need the following lemma.

Lemma 14.7. *The following are equivalent.*

1. $Z_G(\mathbf{w}) > 0$ *for all* $-\mathbf{p} \leq \mathbf{w} \leq 0$.
2. *For any path in* $(-\infty, 0]^X$ *connecting* 0 *and* $-\mathbf{p}$*,* $Z_G > 0$ *on that path.*
3. $Z_G(\mathbf{w}) \neq 0$ *on* $\bar{D}_{\mathbf{p}}$.
4. *For all* $S \subseteq X$*,* $Z_S(-\mathbf{p}) > 0$.
5. *For all* $S \subseteq X$*,* $Z_G(-\mathbf{p}; S)(-1)^{|S|} \geq 0$.

Proof. We will not prove these equivalences here but refer the interested reader to Theorem 2.10 of [SS05] for a full account of the proof. We, however, note in passing that the proof crucially uses the alternating sign property of Mayer coefficients which we will prove later. □

We will also need the following crucial identity.

Proposition 14.8 (Fundamental identity)**.** *Fix $x_0 \in X$. Then*

$$Z_G(\mathbf{w}) = Z_{X \setminus \{x_0\}}(\mathbf{w}) + w_{x_0} Z_{X \setminus \Gamma^*(x_0)}(\mathbf{w}).$$

Proof. The proof uses two simple observations: if $X' \subseteq X$ is an independent set, then

(a) $x_0 \notin X' \Rightarrow X' \subseteq X \setminus \{x_0\}$ and

(b) $x_0 \in X' \Rightarrow X' \setminus \{x_0\} \subseteq X \setminus \Gamma^*(x_0)$.

□

Now we are ready to prove Theorem 14.6.

Proof of Theorem 14.6. "If part." Suppose $\mathbf{p}$ is such that $Z_G \neq 0$ in $\bar{D}_{\mathbf{p}}$. We will show that for any family $(A_x)_{x\in X}$ with dependency graph G we have

$$P(\bigcap_{x\in X} \bar{A}_x) \geq Z_G(-\mathbf{p}) > 0.$$

To do so we will first construct a family $(B_x)_{x\in X}$ with dependency graph G such that $P(\cap_{x\in X}\bar{B}_x)$ is as small as possible, i.e. $Z_G(-\mathbf{p})$. Intuitively, this can be done by making the events B_x as disjoint as possible. With this in mind let us define a probability measure on $\sigma(\{B_x \mid x \in X\})$ as follows:

$$\mathbb{P}(\bigcap_{x\in S} B_x) = \begin{cases} \prod_{x\in S} p_x & \text{if } S \text{ is independent in } G, \\ 0 & \text{otherwise.} \end{cases} \tag{14.1}$$

Note that, by the inclusion-exclusion formula, for all $S \subseteq X$,

$$\begin{aligned} \mathbb{P}(\bigcap_{x\in S} B_x \cap \bigcap_{x\notin S} \bar{B}_x) &= \sum_{S\subseteq T} (-1)^{|T|-|S|}\mathbb{P}(\bigcap_{x\in T} B_x) \\ &= \sum_{\substack{S\subseteq T \\ T \text{ independent}}} (-1)^{|S|} \prod_{x\in T}(-p_x) \\ &= (-1)^{|S|} Z_G(-\mathbf{p}; S). \end{aligned}$$

By Lemma 14.7 part (5) this is indeed nonnegative for all S. Note that if we take $S = \emptyset$, then the above computation reduces to

$$\mathbb{P}(\bigcap_{x\in X} \bar{B}_x) = Z_G(-\mathbf{p}).$$

We now show that $(B_x)_{x\in X}$ is a family minimizing $\mathbb{P}(\cap_{x\in X}\bar{B}_x)$. For $S \subseteq X$ define

$$P_S = \mathbb{P}(\bigcap_{x\in S} \bar{A}_x) \tag{14.2}$$

$$Q_S = \mathbb{P}(\bigcap_{x\in S} \bar{B}_x). \tag{14.3}$$

We will show by induction on $|S|$ that P_S/Q_S is monotone increasing in S. First of all, by inclusion-exclusion,

$$\begin{aligned} Q_S &= \sum_{T\subseteq S} (-1)^{|T|}\mathbb{P}(\bigcap_{x\in T} B_x) \\ &= \sum_{\substack{T\subseteq S \\ T \text{ independent}}} (-1)^{|T|} \prod_{x\in T} p_x \\ &= Z_G(-\mathbf{p}\mathbf{1}_S) = Z_S(-\mathbf{p}) > 0, \end{aligned}$$

by Lemma 14.7 part (4). Also, by the fundamental identity, for $y \notin S$,

$$\begin{aligned} Q_{S\cup\{y\}} &= Z_G(-\mathbf{p}\mathbf{1}_{S\cup\{y\}}) \\ &= Z_G(-\mathbf{p}\mathbf{1}_S) + (-p_y) Z_G(-p\mathbf{1}_{S\setminus\Gamma(y)}) \\ &= Q_S - p_y Q_{S\setminus\Gamma(y)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} P_{S\cup\{y\}} &= \mathbb{P}(\bigcap_{x\in S\cup\{y\}} \bar{A}_x) \\ &= \mathbb{P}(\bigcap_{x\in S} \bar{A}_x) - \mathbb{P}(A_y \cap \bigcap_{x\in S} \bar{A}_x) \\ &\geq P_S - \mathbb{P}(A_y \cap \bigcap_{x\in S\setminus\Gamma(y)} \bar{A}_x) \\ &= P_S - \mathbb{P}(A_y) P_{S\setminus\Gamma(y)} \\ &\geq P_S - p_y P_{S\setminus\Gamma(y)}. \end{aligned}$$

Suppose that the monotonicity property holds for subsets of size $\leq |S|$. Then

$$Q_S P_{S\cup\{y\}} - P_S Q_{S\cup\{y\}} \geq p_y (P_S Q_{S\setminus\Gamma(y)} - Q_S P_{S\setminus\Gamma(y)}) \geq 0,$$

because by induction hypothesis

$$\frac{P_S}{Q_S} \geq \frac{P_{S\setminus\Gamma(y)}}{Q_{S\setminus\Gamma(y)}}.$$

This establishes that P_S/Q_S is monotone increasing in S and thus

$$\frac{P_X}{Q_X} \geq \frac{P_\emptyset}{Q_\emptyset} = 1,$$

which implies that

$$\mathbb{P}(\bigcap_{x\in X} \bar{A}_x) = P_X \geq Q_X = \mathbb{P}(\bigcap_{x\in X} \bar{B}_x) = Z_G(-\mathbf{p}) > 0.$$

"Only if part." If Z_G vanishes somewhere in $\bar{D}_{\mathbf{p}}$, we can choose a minimal $\mathbf{p}'$, $\mathbf{0} \leq \mathbf{p}' \leq \mathbf{p}$ such that $Z_G(-\mathbf{p}') = 0$. Construct a family (B'_x) via the prescription (14.1) with $\mathbf{p}$ replaced by $\mathbf{p}'$. Then clearly

$$\mathbb{P}(\bigcap_{x\in X} \bar{B}'_x) = Z_G(-\mathbf{p}') = 0.$$

(That this defines a proper probability measure follows from the facts that $\mathbf{p}'$ is minimal, $\mathbf{p}'$ belongs to the closure of the set of all good $\mathbf{p}$'s for G, and Z_G is continuous.) So $\mathbb{P}(B'_x) = p'_x \leq p_x$, but

$$\mathbb{P}(\bigcap_{x\in X} \bar{B}'_x) = 0.$$

This completes the proof. □

In order to apply Theorem 14.6 we need easily checkable conditions that guarantee the nonvanishing of Z_G in some polydisc. The following is a corollary to more general results due to Dobrushin [Dob96a, Dob96b], Sokal [Sok01]. Notice that the conditions are the same conditions that we find in Lovász local lemma.

Proposition 14.9. *If there exists* $(r_x)_{x\in X} \in [0,1)^X$ *such that*

$$R_x \leq r_x \prod_{y\in\Gamma(x)} (1-r_y),$$

then for all $\mathbf{w} \in \bar{D}_R$,

$$|Z_G(\mathbf{w})| \geq Z_G(-\mathbf{R}) \geq \prod_{x\in X} (1-r_x) > 0.$$

Proof. The proof uses the fundamental identity in an inductive fashion. In fact, the following more general result holds under our assumptions: for all $S \subseteq T \subseteq X$,

$$\frac{Z_T(\mathbf{w})}{Z_S(\mathbf{w})} \geq \prod_{x\in T\setminus S} (1-r_x). \tag{14.4}$$

To prove this we use induction: suppose that (14.4) holds for all sets strictly smaller than T. Now note that we can write

$$\frac{Z_T(\mathbf{w})}{Z_S(\mathbf{w})} = \frac{Z_T(\mathbf{w})}{Z_{T\setminus\{y\}}(\mathbf{w})} \frac{Z_{T\setminus\{y\}}(\mathbf{w})}{Z_S(\mathbf{w})}. \tag{14.5}$$

Using the fundamental identity the first term on the right can be rewritten as

$$\frac{Z_T(\mathbf{w})}{Z_{T\setminus\{y\}}(\mathbf{w})} = 1 + w_y \frac{Z_{T\setminus\Gamma^*(y)}(\mathbf{w})}{Z_{T\setminus\{y\}}(\mathbf{w})} \tag{14.6}$$

Now by induction hypothesis

$$\frac{Z_{T\setminus\Gamma^*(y)}(\mathbf{w})}{Z_{T\setminus\{y\}}(\mathbf{w})} \geq \frac{1}{\prod_{x\in\Gamma(y)}(1-r_x)}$$

On the other hand, $|w_y| \leq R_y \leq r_y \prod_{x\in\Gamma(y)}(1-r_x)$. Therefore, by (14.6), we have

$$\left|\frac{Z_T(\mathbf{w})}{Z_{T\setminus\{y\}}(\mathbf{w})}\right| \geq 1 - r_y \frac{\prod_{x\in\Gamma(y)}(1-r_x)}{\prod_{x\in\Gamma(y)}(1-r_x)} = 1 - r_y.$$

Now this and another application of the induction hypothesis on the second term of the decomposition (14.5) gives us (14.4). Finally, choosing $T = X$ and $S = \emptyset$ we obtain the desired claim. □

Remark 14.10. If G has maximum degree Δ, set $r_x = \frac{1}{\Delta+1}$. Then we see that $|w_x| \leq \Delta^\Delta/(\Delta+1)^{\Delta+1}$ for all x implies that $Z_G(\mathbf{w}) \geq [\Delta/(\Delta+1)]^{|X|} > 0$. Since $\Delta^\Delta/(\Delta+1)^{\Delta+1} \geq \frac{1}{e\Delta}$, it follows that $\mathbf{p}$, defined by $p_x = \frac{1}{e\Delta}$ for all x, is good for G.

14.2 The Mayer expansion

The partition function of a *repulsive* lattice gas with *fugacity* vector $\mathbf{w} = (w_x)_{x \in X}$ and two-particle Boltzmann factor $W : X \times X \to [0, 1]$, where $W(x, y) = W(y, x)$, is given by

$$Z_W(\mathbf{w}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1,\ldots,x_n \in X} \left(\prod_{i=1}^{n} w_{x_i} \right) \prod_{1 \le i < j \le n} W(x_i, x_j).$$

Since $Z_W(\mathbf{0}) = 1$, $\log Z_W$ is analytic in a neighborhood of $\mathbf{0}$ and thus can be expanded into a convergent power series. This is known as the *Mayer expansion*:

$$\log Z_W(\mathbf{w}) = \sum_{\mathbf{n} \ge \mathbf{0}} c_{\mathbf{n}}(W) \mathbf{w}^{\mathbf{n}}.$$

The coefficients $c_{\mathbf{n}}(W)$ satisfy an interesting alternating sign property:

$$(-1)^{|\mathbf{n}|-1} c_{\mathbf{n}}(W) \ge 0. \tag{14.7}$$

In this section we shall visit a proof (due to Scott and Sokal [SS05]) of this alternating sign property. The first step is to rewrite the partition function in a suitable way:

$$Z_W(\mathbf{w}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1,\ldots,x_n \in X} \left(\prod_{i=1}^{n} w_{x_i} \right) \sum_{G \in \mathcal{G}_n} \prod_{\{i,j\} \in E(G)} (W(x_i, x_j) - 1),$$

where $\mathcal{G}_n$ is the set of all simple graphs on n vertices. This follows from the identity

$$\prod_{1 \le i < j \le n} W(x_i, x_j) = \sum_{G \in \mathcal{G}_n} \prod_{\{i,j\} \in E(G)} (W(x_i, x_j) - 1).$$

Define $\mathcal{W}(G) := \sum_{x_1,\ldots,x_n \in X} \left(\prod_{i=1}^{n} w_{x_i} \right) \prod_{\{i,j\} \in E(G)} (W(x_i, x_j) - 1)$. Then

$$Z_W(\mathbf{w}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{G \in \mathcal{G}_n} \mathcal{W}(G).$$

Note that (i) $\mathcal{W}(\emptyset) = 1$, (ii) for two isomorphic graphs G and G' one has $\mathcal{W}(G) = \mathcal{W}(G')$ and (iii) if $G = G_1 \uplus G_2$ is the disjoint union of two graphs G_1 and G_2, then $\mathcal{W}(G) = \mathcal{W}(G_1)\mathcal{W}(G_2)$. We will now use the *exponential formula* to express $\log Z_G(w)$ as a power series. Let $\mathcal{C}_n$ be the set of all connected graphs on n vertices. Then we have

$$\begin{aligned}
\log Z_G(w) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{G \in \mathcal{C}_n} \mathcal{W}(G) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1,\ldots,x_n \in X} \left(\prod_{i=1}^{n} w_{x_i} \right) \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in E(G)} (W(x_i, x_j) - 1) \\
&= \sum_{\mathbf{n} \ge 0} \left(\frac{1}{n!} \sum_{\substack{x_1,\ldots,x_n \in X \\ \#\{i | x_i = x\} = n_x, \forall x}} \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in E(G)} (W(x_i, x_j) - 1) \right) \mathbf{w}^{\mathbf{n}}.
\end{aligned}$$

Thus

$$c_{\mathbf{n}}(W) = \frac{1}{n!} \sum_{\substack{x_1,\ldots,x_n \in X \\ \#\{i | x_i = x\} = n_x, \forall x}} \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in E(G)} (W(x_i, x_j) - 1).$$

In order to analyze $c_{\mathbf{n}}(W)$, let us introduce the *connected sum*. Let $H = (V, E)$ be a graph, possibly with loops an multiple edges. Let $\mathbf{z} = (z_e)_{e \in E}$ be a complex family of edge weights for H. Then the generating function for the connected spanning subgraphs of G, connected sum for short, is

$$C_H(\mathbf{z}) := \sum_{\substack{E' \subseteq E \\ (V,E') \text{ connected}}} \prod_{\mathbf{e} \in E'} z_e.$$

Then it is easy to verify that C_H satisfies the following deletion-contraction relation:

$$C_H(\mathbf{z}) = C_{H \setminus \{e\}}(\mathbf{z}) + z_e C_{H/e}(\mathbf{z}_{\neq e}).$$

Now we need a concept known as partitionability.

Let $\mathcal{C}$ (resp. $\mathcal{T}$) be the set of subsets $E' \subseteq E$ such that (V, E') is connected (resp. is a tree). Clearly $\mathcal{C}$ is an increasing family of subsets of E with respect to set-theoretic inclusion, and the minimal elements of $\mathcal{C}$ are precisely those of $\mathcal{T}$ (i.e. the spanning trees). Then the (anti-)complex $\mathcal{C}$ is *partitionable*, i.e. there exists a map $\mathsf{R}\colon \mathcal{T} \to \mathcal{C}$ such that $\mathsf{R}(T) \supseteq T$ for all $T \in \mathcal{T}$ and $\mathcal{C} = \biguplus_{T \in \mathcal{T}} [T, \mathsf{R}(T)]$ (disjoint union), where $[E_1, E_2]$ denotes the Boolean interval $\{E' \colon\ E_1 \subseteq E' \subseteq E_2\}$.

Lemma 14.11. *Let $H = (V, E)$ be a connected graph. Then there exists a map $\mathsf{R}\colon \mathcal{T} \to \mathcal{C}$ such that*

(a) $\mathsf{R}(T) \supseteq T$ for all $T \in \mathcal{T}$;

(b) $\mathcal{C}$ is the disjoint union of the Boolean intervals $[T, \mathsf{R}(T)]$, $T \in \mathcal{T}$.

Proof. If H has one vertex and no edges, then $\mathcal{T} = \mathcal{C} = \{\emptyset\}$ and the result holds trivially; so assume henceforth that $E \neq \emptyset$. Assign arbitrary weights $w_e > 0$ chosen so that no two spanning trees have equal weight (for example, one can choose the w_e to be linearly independent over the rationals). For each $E' \in \mathcal{C}$, let $\mathsf{S}(E')$ be the (unique) minimum-weight spanning tree contained in E'. (This can be constructed by a greedy algorithm, i.e. start from $\emptyset$ and keep adding the lowest-weight edge in E' that does not create a cycle.) We then define $\mathsf{R}(T)$ to be the union of all E' that have $\mathsf{S}(E') = T$. To verify that this works, we need to show that if $\mathsf{S}(E_1) = \mathsf{S}(E_2) = T$, then $\mathsf{S}(E_1 \cup E_2) = T$; but this follows easily from the validity of the greedy algorithm. □

Given the existence of R, we have the following simple but fundamental identity:

Proposition 14.12 (partitionability identity)**.** *Let $\mathsf{R}\colon \mathcal{T} \to \mathcal{C}$ be any map satisfying $\mathsf{R}(T) \supseteq T$ for all $T \in \mathcal{T}$ and $\mathcal{C}$ is the disjoint union of the Boolean intervals $[T, \mathsf{R}(T)]$, for $T \in \mathcal{T}$. Then*

$$\begin{aligned} C_H(\mathbf{z}) &= \sum_{\substack{T \subseteq E \\ (V,T) \text{ tree}}} \prod_{e \in T} z_e \sum_{T \subseteq E' \subseteq \mathsf{R}(T)} \prod_{e \in E' \setminus T} z_e \\ &= \sum_{\substack{T \subseteq E \\ (V,T) \text{ tree}}} \prod_{e \in T} z_e \prod_{e \in \mathsf{R}(T) \setminus T} (1 + z_e). \end{aligned} \tag{14.8}$$

This identity (for one specific choice of R) is due originally to Penrose [Pen67].

Proof of the alternating sign property 14.7. We specialize to the Mayer expansion (14.7) by taking $H = K_n$ (where $n = |\mathbf{n}|$ and K_n denotes the complete graph on n vertices), $\mathsf{x} = (x_1, \ldots, x_n)$ and $\mathbf{z}(\mathsf{x})_{ij} = W(x_i, x_j) - 1$. Summing over $x_1, \ldots, x_n \in X$ with the specified cardinalities we get

$$c_{\mathbf{n}}(W) = \frac{1}{n!} \sum_{\substack{x_1,\ldots,x_n \in X \\ \#\{i | x_i = x\} = n_x, \forall x}} C_{K_n}(\mathbf{z}(\mathsf{x})) \tag{14.9}$$

Since we have a repulsive lattice gas, i.e. $W(x, y) \in [0, 1]$, note that $z_{ij} \leq 0$, for all i, j. Therefore for any spanning tree T of K_n

$$(-1)^{n-1} \prod_{\{i,j\} \in T} z(\mathsf{x})_{ij} \geq 0.$$

Also clearly

$$\prod_{\{i,j\} \in \mathsf{R}(T) \setminus T} (1 + z(\mathsf{x})_{ij}) > 0.$$

Therefore, in view of the identity (14.8), we have that $(-1)^{n-1} C_{K_n}(\mathbf{z}(\mathsf{x})) \geq 0$, and a fortiori the representation (14.9) yields

$$(-1)^{|\mathbf{n}|-1} c_{\mathbf{n}}(W) \geq 0.$$

□

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