Lecture 13: Hyperbolic Polynomials in 3 variables are determinants of PSD matrices

Lecturer: J. Kileel *Scribe: Z. Bartha*

In this lecture we study the connection between hyperbolic real homogeneous polynomials (or equivalently, hyperbolic hypersurfaces) and polynomials that have Hermitian determinantal representation, as described in the following definition.

Definition 13.1. A real homogeneous polynomial f of degree d in n variables is said to have a Hermitian determinantal representation if there exist Hermitian $d \times d$ matrices $M_1, M_2, \ldots, M_n$ such that

$$f(x_1, x_2, \ldots, x_n) = \det(x_1 M_1 + x_2 M_2 + \ldots, x_n M_n).$$

The representation is definite if there exists $e \in \mathbb{R}^n$ for which

$$e_1 M_1 + e_2 M_2 + \ldots + e_n M_n > 0,$$

i.e., the matrix on the left-hand side is positive definite.

It is easy to see that if f has a definite (with respect to e) Hermitian determinantal representation with real symmetric matrices, then f is hyperbolic with respect to e. Helton and Vinnikov [HV07] proved the converse of this in three dimensions:

Theorem 13.2 (Helton, Vinnikov)**.** *Let $f \in \mathbb{R}[x, y, z]_d$ be hyperbolic with respect to $e = (e_1, e_2, e_3) \in \mathbb{R}^3$. Then there exist symmetric real matrices A, B and C such that $f = \det(xA + yB + zC)$ and $e_1 A + e_2 B + e_3 C > 0$.*

Corollary 13.3. *Every 3-dimensional hyperbolicity cone is a spectahedron.*

Corollary 13.3 is a special case of the generalized Lax conjecture which states the same for any dimension.

In this lecture, we explain the much simpler proof of Plaumann and Vinzant [PV13] for the following version of the theorem:

Theorem 13.4. *Let $f \in \mathbb{R}[x, y, z]_d$ be hyperbolic with respect to $e = (e_1, e_2, e_3) \in \mathbb{R}^3$. Then there exists a complex Hermitian matrix of linear polynomials $M(x, y, z)$ such that $f(x, y, z) = \det(M(x, y, z))$ and $M(e) > 0$.*

Proposition 13.5. *Corollary 13.3 follows even from Theorem 13.4.*

Proof. Write $M = A + iB$ where A is real symmetric and B is real skew-symmetric. Let

$$N = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

Then N is conjugate to the following matrix

$$\begin{bmatrix} A - iB & 0 \\ 0 & A + iB \end{bmatrix},$$

therefore $\det(N) = f^2$. □

Proof of Theorem 13.4. We will denote the complex (real) hypersurface defined by f by $V_{\mathbb{C}}(f)$ ($V_{\mathbb{R}}(f)$). Given $f \in \mathbb{R}[x, y, z]_d$ hyperbolic with respect to e, assume that $V_{\mathbb{C}}(f)$ is smooth, i.e., $\Delta f \neq 0$ on $V_{\mathbb{C}}(f)$.

First, we construct the adjugate $\mathrm{adj}(M)$. There are three necessary conditions on $\mathrm{adj}(M)$:

1. the entries are in $\mathbb{C}[x, y, z]_{d-1}$,
2. $\mathrm{adj}(M)$ is Hermitian,
3. $M \cdot \mathrm{adj}(M) = f \cdot I$ (thus, $\det(\mathrm{adj}(M)) = f^{d-1}$).

Then, on $V_{\mathbb{C}}(f)$, M will be generically of rank $d - 1$, so $\mathrm{adj}(M)$ is of rank at most 1 generically. This implies that on $V_{\mathbb{C}}(f)$ the 2×2 minors of $\mathrm{adj}(M)$ are 0. Therefore, f divides each 2×2 minors of $\mathrm{adj}(M)$.

Proposition 13.6. *Let A be a 3×3 matrix satisfying the three conditions above. Then every entry of $\mathrm{adj}(A)$ is divisible by f^{d-2} and for $M = \frac{\mathrm{adj}(A)}{f^{d-2}}$ there exists $\gamma \in \mathbb{R}$ such that $\det(M) = \gamma f$.*

Proof (sketch). We can show by induction that each $k \times k$ minor of A is divisible by f^{d-1}. If we take $k = d - 1$, we get that $M = \frac{\mathrm{adj}(A)}{f^{d-2}}$ is well-defined. If we take $k = df^{d-1} |\det(A)|$, then we get that $\det(A) = cf^{d-1}$ for some $c \in \mathbb{R}$. If $c \neq 0$, then

$$\begin{aligned} A \cdot \mathrm{adj}(A) &= cf^{d-1} I \\ A \cdot M &= cfI \\ cf^{d-1} \det(M) &= c^d f^d \\ \det(M) &= c^{d-1} f, \end{aligned}$$

and we let $\gamma = c^{d-1}$. □

Remark 13.7. The scalar γ can be 0. If the polynomials in the top row of A are linearly dependent, then the columns of A are linearly dependent.

Now we give an algorithm that takes f as input and outputs M. This is a modification of Dixon's algorithm [Dix].

1. Set $a_{11} = D_e f$.

2. Compute $V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(D_e f)$. It is the intersection of curves of dimension d and dimension $d-1$ on the projective plane, thus, by Bézout's theorem, it consists of $d(d-1)$ points. The hyperbolicity and smoothness imply that none of these points are real. We write this as $S \cup \overline{S}$ where S consists of $\frac{d(d-1)}{2}$ points.

3. Extend a_{11} to a linearly independent set of degree $d-1$ polynomials vanishing on S and fill in the top row of $A : (a_{11}, a_{12}, \ldots, a_{1d})$. This can be done, because the number of degrees of freedom is $\binom{d+1}{2} - \frac{d(d-1)}{2} = d$. (There are $\binom{d+1}{2}$ degrees of freedom for a $(d+1)$-degree polynomial in three variables.)

4. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ \overline{a_{12}} & a_{22} & \cdots & a_{2d} \\ \overline{a_{13}} & \overline{a_{23}} & \cdots & a_{3d} \\ \vdots & & & \\ \overline{a_{1d}} & \overline{a_{2d}} & \cdots & a_{dd} \end{bmatrix}.$$

For $i \geq 2$, $j \geq 2$ consider the 2×2 minor $a_{11}a_{ij} - a_{1i}\overline{a_{ij}}$. This uniquely determines a_{ij}.

5. Output: $M = f^{2-d}\text{adj}(A)$.

Existence of a_{ij} in step 4. is equivalent to $a_{ii}\overline{a_{ij}} \in < a_{11}, f >$ (the generated ideal by a_{11} and f), which is, by Max Noether's theorem, further equivalent to that the common zeroes in $V_{\mathbb{C}}(a_{1i}\overline{a_{ij}}) \cap V_{\mathbb{C}}(f)$ have multiplicity at least that of the common zeroes in $V_{\mathbb{C}}(a_{11}) \cap V_{\mathbb{C}}(f)$. By the construction, the former intersection includes $S \cup \overline{S}$ (the inclusion of $\overline{S}$ automatically holds).

It remains to show that $\gamma \neq 0$ and that A is positive semi-definite.

□

Finally, we give a condition that implies that the Hermitian matrix in the determinantal representation is positive definite. We will need the following definition.

Definition 13.8. $h \in \mathbb{R}[x]_{d-1}$ interlaces $f \in \mathbb{R}[x]_d$ with respect to e if $h(te + x)$ interlaces $f(te + x)$ for every x.

It is easy to see that the following holds:

Proposition 13.9. *If $f(x, y, z) = \det(xA + yB + zC)$ for some positive semi-definite matrices A, B and C, then f is hyperbolic with respect to $e = (1, 1, 1)$ and $a_{11} = D_e f$ interlaces f with respect to e.*

Now, we will prove the following:

Theorem 13.10. *If $f = \det(M)$ is hyperbolic with respect to e and $D_e f = e_1^T \text{adj}(M) e_1$ interlaces f, then $M(e)$ is positive definite.*

Proof. It suffices to show that $\lambda^T \text{adj} M(e)\overline{\lambda} > 0$ or equivalently that $\lambda^T f(e)(M(e))^{-1}\overline{\lambda}$ for every $\lambda \in \mathbb{C}^d$. Let us fix λ. We will show that $\lambda^T \text{adj}(M)\overline{\lambda}$ interlaces f and that $\lambda^T \text{adj}(M)\overline{\lambda}$ has the same sign at e for every λ.

Lemma 13.11. *If g and h are polynomials of degree $d-1$, f is a univariate polynomial of degree d, g interlaces f and $h(x)g(x) > 0$ whenever $f(x) > 0$, then h interlaces f.*

Proof. Since g and h have the same sign at the roots of f, h changes sign between every pair of roots. □

$\mathrm{adj}M(x)$ has rank 1 for generic $x \in V_{\mathbb{R}}(f)$. Using this for $\lambda_j e_1$, the gram matrix

$$\begin{bmatrix} - & e_1^T & - \\ - & \lambda^T & - \end{bmatrix} \mathrm{adj}(M) \begin{bmatrix} | & | \\ e_1 & \lambda \\ | & | \end{bmatrix}$$

has rank 1. Thus,

$$(e_1^T \mathrm{adj}(M) e_1)(\lambda^T \mathrm{adj}(M)\lambda) - (e_1^T A\overline{\lambda})\overline{(e_1^T A\overline{\lambda})} = 0,$$

and the first term is nonnegative. So $\lambda^T \mathrm{adj}(M)\lambda$ interlaces f by Lemma 13.11, hence, it does not change sign in the component of $\mathbb{R}^n \setminus V_{\mathbb{R}}(f)$ that contains e. Therefore, $(\lambda^T \mathrm{adj}(M)\lambda)(e)$ has the same sign as $(e_1^T \mathrm{adj}(M) e_1)(e)$, which is positive. This finishes the proof of Theorem 13.10.

□

References

[Dix] A. C. Dixon. Note on the reduction of a ternary quantic to a symmetrical determinant. *Cambr. Proc.*, 11:350–351.

[HV07] J William Helton and Victor Vinnikov. Linear matrix inequality representation of sets. *Communications on pure and applied mathematics*, 60(5):654–674, 2007.

[PV13] Daniel Plauman and Cynthia Vinzant. Determinantal representations of hyperbolic plane curves: an elementary approach. *Journal of Symbolic Computation*, 57:48–60, 2013.