

## Lecture 11: Hyperbolic Polynomials and Hyperbolicity Cones

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In this lecture, we will introduce the concept of *hyperbolic polynomials*, a generalization of real stable polynomials. We will also introduce the concept of *hyperbolicity cones*, which are the set of directions along which a polynomial is always real-rooted. We will prove that hyperbolicity cones are convex, study some of their properties, and investigate the connection between barrier arguments and hyperbolicity. It is recommended to look at [Brä] to complement these notes.

## 11.1 Hyperbolic Polynomials

We originally defined real stable polynomials using the following definition.

**Definition 11.1** (Real Stable Polynomials). A nonzero polynomial  $p \in \mathbb{R}[x_1, \dots, x_n]$  is called real stable if and only if it has no zeros in  $\mathcal{H}^n = \{z \in \mathbb{C} : \text{Im}(z) > 0\}^n$ , i.e.,

$$\forall i \quad \text{Im}(x_i) > 0 \implies p(x_1, \dots, x_n) \neq 0.$$

There is an alternate definition of real stability which we will provide in the following lemma.

**Lemma 11.2.** A nonzero polynomial  $p \in \mathbb{R}[x_1, \dots, x_n]$  is real stable if and only if for all  $x \in \mathbb{R}^n$  and  $e \in \mathbb{R}_{>0}^n$  the polynomial  $p(x + te) \in \mathbb{R}[t]$  is real-rooted.

*Proof.* First assume that  $p$  is real stable but  $p(x + te)$  is not real-rooted for some choice of  $x$  and  $e$ . Note that  $p(x + te)$  has real coefficients, so its complex roots come in conjugate pairs. In particular if it has a complex root, it has one in the upper half plane. So we can choose  $t$  such that  $p(x + te) = 0$  and  $\text{Im}(t) > 0$ . But note that  $x + te$  is a vector whose coordinates have positive imaginary parts. This contradicts  $p$  being real stable.

Now assume that  $p(x + te)$  is real-rooted for all valid choices of  $x$  and  $e$ . If  $p$  is not real stable it means that it has a root  $z = (z_1, \dots, z_n) \in \mathcal{H}^n$ . Now let  $x_i = \text{Re}(z_i)$  and  $e_i = \text{Im}(z_i)$ . Then  $x \in \mathbb{R}^n$  and  $e \in \mathbb{R}_{>0}^n$  and  $z = x + iy$ . This means that the polynomial  $p(x + te)$  has a root  $t = i$  which contradicts its being real-rooted.  $\square$

A geometric way of interpreting this new definition, is to visualize  $p(x + te)$  as the restriction of  $p$  on a line originating from  $x$  and being parallel to  $e$ . Then  $p$  being real stable is equivalent to all such one-dimensional restrictions being real-rooted whenever  $e \in \mathbb{R}_{>0}^n$ . This geometric view inspires a generalization to arbitrary  $e$ , which brings us to hyperbolic polynomials.

**Definition 11.3** (Hyperbolic Polynomials). A homogeneous polynomial  $p \in \mathbb{R}[x_1, \dots, x_n]$  is called hyperbolic in direction  $e \in \mathbb{R}^n$  if and only if

$$p(e) > 0, \text{ and} \\ \forall x \in \mathbb{R}^n \quad p(x + te) \in \mathbb{R}[t] \text{ is real rooted.}$$

Before we continue it is worth mentioning a few remarks about this definition.

**Remark 11.4.** Instead of  $p(x + te)$ , sometimes in the literature, alternative one-dimensional restrictions appear in definition 11.3:  $p(x - te)$ ,  $p(te - x)$ , and  $p(e + tx)$ . Real-rootedness of these polynomials and the polynomial  $p(x + te)$  is equivalent since one can translate between the roots of these three polynomials using the maps  $z \mapsto -z$  and  $z \mapsto 1/z$  which leave the real line intact.

**Remark 11.5.** Requiring  $p$  to be homogeneous is a technical requirement, needed for this definition to work. The definition can be extended to non-homogeneous polynomials. For details see [Gül97].

**Remark 11.6.** The condition that  $p(e) > 0$  is sometimes replaced by  $p(e) \neq 0$ . Whenever  $p(e) \neq 0$ , either  $p$  or  $-p$  is positive at  $e$ , so these two conditions are not very different. However one cannot drop the condition that  $p(e) \neq 0$ .

Let us see a few examples of hyperbolic polynomials.

**Example 11.7.** The polynomial  $p(x_1, \dots, x_n) = x_1 x_2 \dots x_n$  is hyperbolic in direction  $e = (1, \dots, 1)$ . The roots of  $p(x + te)$  are exactly  $-x_1, -x_2, \dots, -x_n$ .

**Example 11.8.** The polynomial  $p(x_0, x_1, \dots, x_n) = x_0^2 - x_1^2 - \dots - x_n^2$  is hyperbolic in direction  $e = (1, 0, \dots, 0)$ . The roots of  $p(x + te)$  are  $-x_0 \pm \sqrt{x_1^2 + \dots + x_n^2}$  which are real.

**Example 11.9.** Perhaps the most important example of hyperbolic polynomials is the determinant. Consider the space  $\text{Sym}_n(\mathbb{R})$  of  $n \times n$  symmetric matrices with real entries. The function

$$\det : \text{Sym}^n(\mathbb{R}) \rightarrow \mathbb{R}$$

can be considered as a polynomial in the entries on or above the diagonal. The polynomial  $\det$  is hyperbolic in direction  $e = I$ , the identity matrix. The roots of  $p(X + tI)$  are simply the negative of the eigenvalues of  $X$  which are all real since  $X$  is symmetric.

The last example inspires one to generalize many definitions coming from linear algebra to arbitrary hyperbolic polynomials. For example, one can define the notion of eigenvalues for a point  $x$  as the set of roots of  $p(te - x)$ . For more examples see [BGLS01].

## 11.2 Properties of Hyperbolic Polynomials

In this section we explore some operations defined on hyperbolic polynomials along with their properties. First let us make the connection between hyperbolic polynomials and real stable polynomials more formal. For this we need to recall the definition of homogenization.

**Definition 11.10** (Homogenization). Given a polynomial  $p \in \mathbb{R}[x_1, \dots, x_n]$ , define  $p_H \in \mathbb{R}[x_0, x_1, \dots, x_n]$ , its homogenization, in the following way

$$p_H(x_0, x_1, \dots, x_n) = x_0^d \cdot p\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right),$$

where  $d$  is the degree of  $p$ .

Now the connection between real stable and hyperbolic polynomials can be formalized according to the following theorem.

**Theorem 11.11.** *The polynomial  $p \in \mathbb{R}[x_1, \dots, x_n]$  is real stable if and only if  $p_H$  is hyperbolic in direction  $(0, e)$  for all  $e \in \mathbb{R}_{>0}^n$ .*

*Proof.* First assume that  $p$  is real stable. Assume that  $e \in \mathbb{R}_{>0}^n$  and we have a point  $(x, y) \in \mathbb{R} \times \mathbb{R}^n$ . We need to show that  $p_H((x, y) + t(0, e))$  is real-rooted. We have

$$p_H(x, y + te) = x^d p\left(\frac{y_1 + te}{x}, \dots, \frac{y_n + te}{x}\right).$$

If  $x \neq 0$ , then the roots of this polynomial are simply the roots of  $p\left(\frac{y}{x} + te\right) \in \mathbb{R}[t]$  scaled by  $x$ , and therefore they are real. By taking the limit as  $x \rightarrow 0$ , and using continuing of roots, one can see that the roots remain real for  $x = 0$  as well.

Now assume that  $p_H$  is hyperbolic in direction  $(0, e)$  for every  $e \in \mathbb{R}_{>0}^n$ . We need to show that  $p(y + te) \in \mathbb{R}[t]$  is real-rooted for every  $y \in \mathbb{R}^n$ . Consider the univariate restriction  $p_H((1, y) + t(0, e))$ . We have

$$p_H((1, y) + t(0, e)) = p(y + te).$$

So by hyperbolicity of  $p_H$  in direction  $(0, e)$ , we have that  $p(y + te)$  is real-rooted. That finished the proof.  $\square$

A very useful operation that preserves hyperbolicity is differentiation.

**Definition 11.12** (Directional Derivative). Given a polynomial  $p \in \mathbb{R}[x_1, \dots, x_n]$  and a vector  $v \in \mathbb{R}^n$  define the directional derivative of  $p$  in direction  $v$  as

$$D_v p := \sum_i v_i \frac{\partial p}{\partial x_i}.$$

Starting with a hyperbolic polynomial, one can apply the directional derivative operator to get a new hyperbolic polynomial as the following theorem shows.

**Theorem 11.13.** *If  $p \in \mathbb{R}[x_1, \dots, x_n]$  is hyperbolic in direction  $e \in \mathbb{R}^n$ , then  $D_e p$  is hyperbolic in direction  $e$  as well, unless  $D_e p = 0$ .*

*Proof.* This is essentially an application of the Rolle's theorem after observing the following:

$$(D_e p)(x + te) = \frac{d}{dt} p(x + te).$$

Since the univariate restriction  $p(x + te)$  has real roots, by Rolle's theorem, its derivative is real-rooted as well; between any two consecutive roots of  $p$  there is one root of the derivative.

One only needs to verify that  $(D_e p)(e) > 0$ . But we have  $p(te) = t^d p(e)$  where  $d$  is the degree of  $p$ . If  $d = 0$ , then  $D_e p = 0$ , so assume that  $d \geq 1$ . Now

$$(D_e p)(te) = \frac{d}{dt} t^d p(e) = dt^{d-1} p(e),$$

which means that  $(D_e p)(e) = d \cdot p(e) > 0$ . □

Theorem 11.13 is powerful, since it can be used to produce new examples of hyperbolic polynomials. Let us see some applications of this theorem.

**Example 11.14.** Starting with the polynomial  $p(x) = x_1 \dots x_n$  from example 11.7, after applying  $D_e$  many times with  $e = (1, \dots, 1)$ , one gets constant multiples of the elementary symmetric polynomials:

$$e_k(x_1, \dots, x_n) = \sum_{S \subseteq \binom{[n]}{k}} \prod_{i \in S} x_i \quad (0 \leq k \leq n).$$

**Example 11.15.** Starting with the polynomial  $p(X) = \det(X)$  defined on  $\text{Sym}^n(\mathbb{R})$ , after many applications of  $D_e$  with  $e = I$ , one gets the sum of  $k$ -minors (for an appropriate  $k$ ) defined as follows:

$$\sigma_k(X) = \sum_{S \subseteq \binom{[n]}{k}} \det(X_{S,S}),$$

where  $X_{S,S}$  is the submatrix obtained from rows  $S$  and columns  $S$ . The polynomials  $\sigma_k$  can also be obtained by feeding the eigenvalues of  $X$  into the elementary symmetric polynomials from example 11.14:

$$\sigma_k(X) = e_k(\lambda_1(X), \dots, \lambda_n(X)),$$

where  $\lambda_1(X) \leq \dots \leq \lambda_n(X)$  are the eigenvalues of  $X$ . It is easy to observe that the values  $\sigma_k(X)$  are (up to signs) the coefficients of the characteristic polynomial  $\det(\lambda I - X)$ .

We now know a few ways to construct hyperbolic polynomials. Another, often overlooked, way of constructing hyperbolic polynomials is restriction to linear subspaces. We will use the following fact later.

**Fact 11.16.** Assuming  $p \in \mathbb{R}[x_1, \dots, x_n]$  is hyperbolic in direction  $e$  and  $V \subseteq \mathbb{R}^n$  is a linear subspace containing  $e$ , the restriction  $p|_V : V \rightarrow \mathbb{R}$  is also hyperbolic in direction  $e$ .

### 11.3 Hyperbolicity Cone

To each hyperbolic polynomial one can assign a geometric object called the hyperbolicity cone.

**Definition 11.17** (Hyperbolicity Cone). Let  $p \in \mathbb{R}[x_1, \dots, x_n]$  be hyperbolic in direction  $e$ . Define the hyperbolicity cone of  $p$  w.r.t.  $e$  as

$$K(p, e) = \{x \in \mathbb{R}^n : p(x - te) \in \mathbb{R}[t] \text{ has positive roots}\}.$$

Note that the name “cone” is justified because the polynomials we consider are homogeneous.

**Lemma 11.18.** For a homogeneous polynomial  $p(x_1, \dots, x_n)$  hyperbolic in direction  $e$ , the set  $K(p, e)$  is a cone.

*Proof.* If  $x \in K(p, e)$  and  $c > 0$ , then  $cx \in K(p, e)$ , because of the following identity:

$$p(cx - te) = c^d \cdot p\left(x - \frac{t}{c}e\right).$$

The roots of  $p(cx - te)$  and  $p(x - te)$  are just a factor of  $c$  different from each other. In particular their signs are the same.  $\square$

Before studying the properties of hyperbolicity cones, let us first see some examples of them.

**Example 11.19.** Given the polynomial  $p(x) = x_1 \dots x_n$  which is hyperbolic in direction  $e = (1, \dots, 1)$ , the hyperbolicity cone is the positive orthant:

$$K(p, e) = \{x \in \mathbb{R}^n : \forall i x_i > 0\}.$$

This is because the roots of  $p(x - te)$  are simply the coordinates of  $x$ .

**Example 11.20.** Given the polynomial  $p(x) = \det(X)$  defined on  $\text{Sym}^n(\mathbb{R})$  with direction  $e = I$ , the hyperbolicity cone is the PSD cone:

$$K(p, e) = \{X \in \text{Sym}^n(\mathbb{R}) : X \succ 0\}.$$

This is because the roots of  $p(X - te)$  are simply the eigenvalues of  $X$ .

Note that linear transformations and linear restrictions respect hyperbolicity as long as the direction of hyperbolicity is preserved. This means that linear sections of the previous cones that contain the respective directions of hyperbolicity are also hyperbolicity cones.

One of the main properties of hyperbolicity cones is their convexity which we will prove next. Convexity opens up the possibility of convex analysis and convex programming over these geometric objects. Convex programming over hyperbolic cones is an interesting subject which is actively researched; see [Gül97] for more details.

**Theorem 11.21.** Assume that  $p \in \mathbb{R}[x_1, \dots, x_n]$  is hyperbolic in direction  $e$ . The following are true regarding  $K(p, e)$ .

1. The set  $K(p, e)$  is the connected component of  $\mathbb{R}^n \setminus \{x : p(x) = 0\}$  that contains  $e$ .

2. For any  $v \in K(p, e)$ ,  $p$  is hyperbolic in direction  $v$ .
3. For any  $v \in K(p, e)$ ,  $K(p, v) = K(p, e)$ .
4. The set  $K(p, e)$  is convex.

*Proof of 1.* We have  $p(e - te) = (1 - t)^d p(e)$ , where  $d$  is the degree of  $p$ . This polynomial has positive roots, therefore  $e \in K(p, e)$ .

Assume that  $C$  is the connected component of  $\mathbb{R}^n \setminus \{x : p(x) = 0\}$  that contains  $e$ . We will first show that  $C \subseteq K(p, e)$ .

Given a point  $x \in C$ , one can connect  $x$  and  $e$  using a path  $y : [0, 1] \rightarrow \mathbb{R}^n$ , where  $y(0) = e$  and  $y(1) = x$ . Now consider the roots of the polynomial  $p(y(\theta) - te) \in \mathbb{R}[t]$  as functions of  $\theta$ . They change continuously; they also never become zero, since  $p(y(\theta)) \neq 0$  (we are moving inside of  $C$ ). This means that the roots never change sign; therefore the roots of  $p(x - te)$  are all positive, which means that  $x \in K(p, e)$ .

Next, we prove that  $K(p, e) \subseteq C$ . It is enough to prove that  $K(p, e)$  is path-connected. Take a point  $x \in K(p, e)$ ; we will construct a path connecting  $x$  to  $e$ . Note that  $x + ce \in K(p, e)$  for all  $c \geq 0$ . This is because the roots of  $p(x + ce - te)$  are just the roots of  $p(x - te)$  added with  $c$ . By lemma 11.18, we also have  $p(\frac{x+ce}{1+c}) \in K(p, e)$ . By varying  $c$  from 0 to  $+\infty$ , one can get every point on the line segment joining  $x$  and  $e$  as  $\frac{x+ce}{1+c}$ . This means that the line segment joining  $x$  and  $e$  is inside  $K(p, e)$ .  $\square$

*Proof of 2.* Let  $v \in K(p, e)$  and  $x \in \mathbb{R}^n$ . We will prove that for any  $\alpha, \beta > 0$ , the roots of the polynomial  $p(\beta x - tv + i\alpha e) \in \mathbb{R}[t]$  lie in the upper half plane  $\mathcal{H}$ . By taking the limit as  $\alpha \rightarrow 0$ , and letting  $\beta = 1$ , one can see that  $p(x - tv)$  has no roots in the lower half plane. Therefore it must be real-rooted.

Now let us prove that the roots of  $p(\beta x - tv + i\alpha e)$  are in  $\mathcal{H}$ . First let us prove this for  $\beta = 0$ . If  $p(-tv + i\alpha e) = 0$ , then

$$(-t)^d \cdot p(v - \frac{i\alpha}{t}e) = 0,$$

where  $d$  is the degree of  $p$ . Clearly  $t$  cannot be 0 because  $p(e) \neq 0$ . So  $p(v - \frac{i\alpha}{t}e) = 0$ . But this means that  $\frac{i\alpha}{t} \in \mathbb{R}_{>0}$  since the roots of  $p(v - te) \in \mathbb{R}[t]$  are all positive reals. Now simply note that  $\frac{i\alpha}{t} \in \mathbb{R}_{>0}$  means  $t = ci$  for some  $c > 0$ , which means that  $t \in \mathcal{H}$ .

Now assume that for some  $\beta > 0$ ,  $p(\beta x - tv + i\alpha e)$  has a root not in  $\mathcal{H}$ . If we take the infimum over all such  $\beta$ , by continuity of roots, one can see that  $p(\beta x - tv + i\alpha e)$  must have a real root (the border of  $\mathcal{H}$ ). So for some  $t \in \mathbb{R}$ , we have  $p(\beta x - tv + i\alpha e) = 0$ . Note that  $\beta x - tv \in \mathbb{R}^n$ , so the roots of  $p((\beta x - tv) + se)$  must be real. This means that  $s = i\alpha$  cannot be a root, which finished the proof.  $\square$

*Proof of 3.* This is obvious from parts 1 and 2.  $\square$

*Proof of 4.* We already saw in the proof of part 1 that the line segment joining any  $v \in K(p, e)$  and  $e$  is inside  $K(p, e)$ . Now if  $w$  is an arbitrary point in  $K(p, e)$ , we have  $K(p, e) = K(p, w)$  by part 3. Therefore  $v \in K(p, w)$ , and the line segment joining  $v$  and  $w$  lies inside  $K(p, w) = K(p, e)$ , which finished the proof.  $\square$

The cones of hyperbolicity generalize the SDP cone as seen by example 11.20. One of the main research questions in the area of hyperbolic polynomials is whether this generalization is strict. The next lecture will explore this further. For more details see the section on generalized Lax conjecture in [Brä].

## 11.4 Connections to the Barrier Method

In the proof of the Kadison-Singer problem, [MSS15], we used barrier functions defined as

$$\phi_i(p) = \partial_i \log(p) = \frac{\partial_i p}{p},$$

where  $p$  was a real stable polynomial. One of the key properties used was the convexity of this function above the roots. This is more general and true for hyperbolic polynomials. Next we will prove a stronger version of this.

**Theorem 11.22.** *Assume that  $p \in \mathbb{R}[x_1, \dots, x_n]$  is hyperbolic in direction  $e \in \mathbb{R}^n$ . For  $v \in K(p, e)$  define the barrier in direction  $v$  as*

$$D_v \log(p) = \frac{D_v p}{p}.$$

*The reciprocal of this function  $\frac{p}{D_v p}$  is concave on  $K(p, e)$ .*

*Proof.* Consider a new variable  $y$  and define

$$H_v(x, y) = p(x) - y \cdot D_v p(x) = (I - y D_v) p(x).$$

Then,  $H_v$  is hyperbolic in direction  $(v, 0)$ . This is because for the univariate restrictions we have

$$H_v((x, y) + t(v, 0)) = p(x + tv) - y \frac{d}{dt} p(x + tv).$$

Note that  $p(x + tv) \in \mathbb{R}[t]$  is real-rooted and therefore  $\frac{d}{dt} p(x + tv)$  changes signs on these roots. Therefore  $p(x + tv) - y \frac{d}{dt} p(x + tv)$  changes signs on the roots of  $p(x + tv)$  and by the intermediate value theorem, it must have real roots.

Next, note that if  $w \in K(p, e) = K(p, v)$ , then  $(w, 0) \in K(H_v, (v, 0))$ . To show this one needs to consider the roots of  $H_v((w, 0) - t(v, 0)) = p(w - tv)$ , which are all positive by assumption. Therefore  $K(p, v) \times \{0\} \subseteq K(H_v, (v, 0))$ .

Now assume that  $w \in K(p, v)$  and that  $H_v(w, y) > 0$ . Then we claim that  $(w, y) \in K(H_v, (v, 0))$ . This is because  $K(H_v, (v, 0))$  is the connected component of space containing  $(v, 0)$  and  $(w, 0)$ , after

removing the zeros of  $H_v$ . But the value of  $H_v$  on the line segment joining  $(w, 0)$  and  $(w, y)$  remains positive, as  $H_v$  is linear on this restriction and is positive at both ends. Since this line segment joins  $(w, 0)$  to  $(w, y)$ , and never crosses a root of  $H_v$ , we have that  $(w, y) \in K(H_v, (v, 0))$ .

The last paragraph implies that the epigraph of the function  $w \mapsto \frac{p(w)}{D_v p(w)}$  over the domain  $K(p, e)$  is contained in  $K(H_v, (v, 0))$ , i.e.

$$\left\{ (w, y) : w \in K(p, v) \text{ and } \frac{p(w)}{D_v p(w)} > y \right\} \subseteq K(H_v, (v, 0)).$$

Now assume that  $y_1 < \frac{p(x_1)}{D_v p(x_1)}$  and  $y_2 < \frac{p(x_2)}{D_v p(x_2)}$ , where  $x_1, x_2 \in K(p, e)$ . Simply note that these imply  $(x_1, y_1) \in K(H_v, (v, 0))$  and  $(x_2, y_2) \in K(H_v, (v, 0))$ . Therefore by taking a convex combination, we have that  $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in K(H_v, (v, 0))$ . But the sign of  $H_v$  over  $K(H_v, (v, 0))$  is constant (positive). This means that

$$\frac{p(\lambda x_1 + (1 - \lambda)x_2)}{D_v p(\lambda x_1 + (1 - \lambda)x_2)} > \lambda y_1 + (1 - \lambda)y_2.$$

Taking the limit by letting  $y_1 \rightarrow \frac{p(x_1)}{D_v p(x_1)}$  and  $y_2 \rightarrow \frac{p(x_2)}{D_v p(x_2)}$  finishes the proof.  $\square$

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