The Open Mapping Theorem

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Theorem. If D is a domain and f is analytic and nonconstant in D then the image $f(D)$ is an open set.

Proof. Suppose $z_0 \in D$ and $w_0 = f(z_0)$. We need to show that there is a neighborhood $D(w_0, \delta)$ such that

$$D(w_0, \delta) \subset f(D),$$

i.e., that for every w with $|w - w_0| < \delta$ there is some $z \in D$ such that $w = f(z)$.

The idea behind the proof is to observe that we are looking for zeros of the function $f(z) - w$ for various w close to w_0, and we already know that the function $f(z) - w_0$ has a zero – namely z_0. So if we can find a contour on which $f(z) - w_0$ is large, we can apply Rouche's theorem to say that for w sufficiently close to w_0, the function $f(z) - w$ must have at least as many zeros inside it as $f(z) - w_0$ does, which is at least one, and we will be done.

Step 1: Observe that there must be a circle $C_\epsilon = \{z : |z - z_0| = \epsilon\}$ centered at z_0 such that $f(z) - w_0 \neq 0$ for all $z \in C_\epsilon$. If there wasn't we would have a zero $z_\epsilon \in C_\epsilon$ of $f(z) - w_0$ on every circle C_ϵ for all $\epsilon > 0$. But then the sequence $z_{1/n}$ would be a sequence of zeros of $f(z) - w_0$ with a limit point, namely z_0, which would imply that $f(z) - w_0$ is identically zero in D, which is impossible since f is not constant. Moreover, note that we can without loss of generality choose the circle C_ϵ to lie entirely in D.

Step 2: Since the function $|f(z) - w_0|$ is continuous on C_ϵ and C_ϵ is compact it must achieve its minimum there; moreover the minimum must be some $\delta > 0$ since $f(z) - w_0 \neq 0$ for $z \in C_\epsilon$.

Step 3: We thus have $|f(z) - w_0| > \delta$ on C_ϵ. Consequently, whenever $|w - w_0| < \delta$, Rouche's theorem tells us that

$$f(z) - w = f(z) - w_0 + (w_0 - w)$$

has the same number of zeros inside C_ϵ as

$$f(z) - w_0.$$

But the latter function has at least once zero inside C_ϵ, so we are done. $\square$