Math 185 Fall 2015, Midterm 2 Solutions

1. True or false (no need for justification). 3 points each.

   (a) If f has an antiderivative in a domain D then f is analytic in D.
   TRUE. If $f(z) = F'(z)$ in D, then F is infinitely differentiable so in particular $f'(z) = F''(z)$ exists in D.

   (b) If f is analytic in a domain D then it has an antiderivative in D.
   FALSE. $f(z) = 1/z$ is analytic in $\mathbb{C} \setminus \{0\}$ but has no antiderivative there (otherwise its integral would be zero).
   Note that this is true if D is simply connected (which is the Cauchy-Goursat theorem).

   (c) The function $f(z) = z^2/\sin(z)$ has a pole at $z = 0$.
   FALSE. The reason is that $\sin(z)$ has a simple zero at 0, whereas z^2 has a zero of order two, so the ratio has a simple zero at 0. More concretely, the limit of $f(z)$ as $z \to 0$ is 0.

   (d) If f and g are entire and agree on infinitely many points then they must be identical.
   FALSE. Consider the functions $\sin(z)$ and $2\sin(z)$ which agree on $z = n\pi$ for integers n, but are not the same function.
   The statement is however true if the two functions agree on a sequence of points with a limit point in the domain in which they are analytic — this is the identity theorem.

   (e) Every entire function has at least one zero.
   FALSE. Consider e^z or any nonzero constant function.

   (f) There is a function $f(z)$ such that f is analytic in $\mathbb{C} \setminus \{0\}$, has a simple pole at 0, and $\oint_{|z|=1} f(z)dz = 0$.
   FALSE. The integral is equal to the residue of $f(z)$ at 0 since there are no other singularities. Since f has a simple pole at zero, its Laurent series is of the form

   $$f(z) = \frac{b_1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

   with $b_1 \neq 0$, so the integral cannot be zero.

   (g) If f and g are two branches of $\log(z)$ analytic in a domain D, then $f - g$ must be constant in D.
   TRUE. All branches of $\log(z)$ have derivative $1/z$ (applying the chain rule to the definition $e^{\log(z)} = z$), so $(f(z) - g(z))' = 1/z - 1/z = 0$ and $f - g$ must be constant.

2. (7 points) Evaluate the integral

   $$\oint_{|z-3|=1} \frac{\cos(z)}{z(z-\pi)^2} dz$$

   oriented positively.

   The integrand $f(z)$ has singularities at $z = 0$ and $z = \pi$; of these only the latter is contained in the circle $|z - 3| = 1$, so by the residue theorem we have

   $$\oint_{|z-3|=1} \frac{\cos(z)}{z(z-\pi)^2} dz = 2\pi i \text{Res}(f, \pi).$$

Since $\cos(\pi) = -1 \neq 0$, this is a pole of order two, and the residue is given by

$$\lim_{z\to\pi} \frac{1}{1!}\frac{d}{dz}(z-\pi)^2\frac{\cos(z)}{z(z-\pi)^2} = \left((\cos(z))(-1/z^2) + (-\sin(z))(1/z)\right)\Big|_{z=\pi} = \frac{-1}{-\pi^2} + 0.$$

Thus, the integral must be equal to $\frac{2i}{\pi}$.

3. (7 points) Find the Taylor expansion of $f(z) = \frac{1}{z}$ at $z_0 = i + 1$. What is its radius of convergence?

   Letting $z_0 = i + 1$, we expand:

$$f(z) = \frac{1}{(z-z_0)+z_0} = \frac{1}{z_0}\frac{1}{1+\frac{z-z_0}{z_0}} = \frac{1}{z_0}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-z_0}{z_0}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n(z-(i+1))^n}{(i+1)^{n+1}},$$

   convergent whenever $\left|\frac{z-z_0}{z_0}\right| < 1$, which is equivalent to $|z-(i+1)| < |i+1| = \sqrt{2}$. Thus, the radius of convergence is $\sqrt{2}$. This can also be seen directly by noting that $f(z)$ is analytic except for a simple pole at $z = 0$, and the distance of this point from $z_0 = 1+i$ is $|z_0 - 0| = \sqrt{2}$.

4. (8 points) Evaluate the integral

$$\oint_{|z|=1}(z+1)^2 e^{3/z^2}\,dz,$$

   oriented positively.

   Since e^{3/z^2} is analytic in $\mathbb{C}\setminus\{0\}$ and $(z+1)^2$ is a polynomial, the integrand $f(z)$ is analytic in $\mathbb{C}\setminus\{0\}$, so by the residue theorem the integral is equal to $2\pi i\text{Res}(f,0)$. We calculate this residue by computing its Laurent series at zero, by multiplying $(z+1)^2$ by the Laurent series for e^{3/z^2}:

$$f(z) = (z^2+2z+1)\left(1 + \frac{3}{z^2} + \frac{1}{2!}\frac{3^2}{z^4} + \dots\right).$$

   Since we are only interested in the coefficient of $1/z$, we do not care about terms in the expansion of e^{3/z^2} of order higher than $1/z^3$ (since multiplying by a quadratic polynomial can only increase the degree by two). Multiplying out the first few terms, we have

$$f(z) = z^2(1+3/z^2+\dots) + 2z(1+3/z^2+\dots) + (1+\dots) = z^2 + 4 + 2z + 6/z + \dots,$$

   so the residue is 6 and the integral has value $12\pi i$.

5. (7 points) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a function with $f(1/n) = 1$ for all positive integers n and $f(i) = 2$. Show that f cannot be entire.

   Assume that f is entire, and observe that the sequence $z_n = 1/n$ has a limit point $\lim_{n\to\infty} z_n = 0$ contained in the domain $\mathbb{C}$ in which f is analytic. Now observe that the constant function $g(z) = 1$ is also analytic in $\mathbb{C}$ and $f(z_n) = g(z_n)$ for all n. By the identity theorem, we must have $f(z) = g(z) = 1$ for all $z \in \mathbb{C}$[1], contradicting that $f(i) = 2$. Thus f cannot be entire.

[1] Recall that this happens because $f - g$ must be analytic in $\mathbb{C}$, $f(0) - g(0) = 0$ by continuity, and 0 is now a non-isolated zero of $f - g$ so $f - g$ must be identically zero.

6. Extra credit (10 points): Assume that C is a simple closed contour. Show that

$$\oint_C \frac{1}{p(z)} dz = 0$$

whenever $p(z)$ is a polynomial of degree at least two with all zeros contained inside C.

Let C_R be a circle of radius R containing C in its interior. Since $p(z) \neq 0$ in the region between C_R and C, we know that $1/p(z)$ is analytic in this region and by the deformation theorem (or by contour splicing) we have

$$I = \oint_C \frac{1}{p(z)} dz = \oint_{C_R} \frac{1}{p(z)} dz.$$

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots a_0$ and choose R large enough so that $|p(z)| > a_n |z|^n/2$ for $|z| \geq R$. The ML estimage implies that

$$|I| \leq 2\pi R \cdot \max_{z \in C_R} \frac{1}{|p(z)|} \leq 2\pi R \cdot \frac{1}{a_n R^n/2} = \frac{4\pi}{a_n R^{n-1}}$$

. Since $n > 1$ this quantity goes to zero as $R \to \infty$, so the integral must be zero.