Math 185 Fall 2015, Sample Midterm 2

50 minutes, one page of notes allowed

1. (3 points each) True or False (no need for justification):

   (a) If $f(z)$ is a polynomial then its Taylor series about any point has only finitely many terms.

   TRUE. Only finitely many derivatives at any point are nonzero, so only finitely many coefficients are nonzero.

   (b) If $f(z) = p(z)/q(z)$ for polynomials p and q, then its Laurent series about any of its singularities has only finitely many terms.

   FALSE. Consider $f(z) = 1/(z-1)(z-2)$ about $z_0 = 1$.

   (c) If f is analytic in $\mathbb{C}$ except for a finite number of removable singularities, then the Taylor series of f at the origin converges in the whole complex plane.

   TRUE. Let the removable singularities be $z_1, \dots, z_k$. Consider the function $g(z)$ which is equal to $f(z)$ wherever it is analytic and defined as $g(z_i) = \lim_{z \to z_i} f(z_i)$ at the removable singularities (this is just a formal way of "removing" the singularities). Then $g(z)$ is entire and has the same Taylor series at the origin as $f(z)$ (by considering a neighborhood of the origin that excludes all singularities), so this series must converge everywhere.

   In general, the point is that removable singularities have no effect on Taylor series.

   (d) If f is nonconstant and entire then

   $$\max_{|z| \le 1} |f(z)| < \max_{|z| \le 2} |f(z)|.$$

   TRUE. Since f is analytic on $D(0,2)$ the maximum modulus principle implies that it has no maximum in $D(0,2)$, or in other words it achieves its maximum on $\overline{D}(0,2)$ on the boundary. Thus,

   $$\max_{|z| < 2} |f(z)| < \max_{|z| \le 2} |f(z)|,$$

   which implies the claim.

   (e) There is a nonconstant entire function such that $|f(z)| < 1 + \sqrt{|z|}$ for all $z \in \mathbb{C}$.

   FALSE. By Cauchy's estimate, we have for every n:

   $$f^{(n)}(0) \le \frac{n! R^{1/2}}{R^n}.$$

   Letting $R \to \infty$ we see that all derivatives are zero, so f must be constant.

2. (10 points) Show that there is no function analytic in $D(0,1)$ which agrees with the function $f(x) = x^2 \sin(1/x)$ on the interval $(0,1)$.

   Assume there was such a function, say $g(z)$. Note that $g(z)$ is in particular analytic in the punctured disk $D^o(0,1)$, and agrees with the function $f(z) = z^2 \sin(1/z)$ which is also analytic in $D^o(0,1)$ on the line segment $(0,1)$. By the identity theorem we must have $g(z) = f(z)$ in $D^o(0,1)$. However, notice that $f(z)$ has an essential singularity (by inspecting its Laurent series) at 0, so $\lim_{z\to 0} f(z) = \lim_{z\to 0} g(z)$ does not exist. Thus $g(z)$ cannot be continuous at 0, and therefore cannot be analytic in $D(0,1)$.

   Another solution: Assume such a function $g(z)$ exists. If $g(z) = 0$ then 0 would be a non-isolated zero of g since $g(1/n\pi) = 0$ for all n, which is impossible since $g(z)$ is not identically zero. If $g(z) \neq 0$ then g cannot be continuous at zero, since we have $\lim_{n\to\infty} g(x) = 0$ for the sequence $x_n = 1/n\pi$ but $g(0) \neq 0$. Thus no such function exists.

3. (10 points) Locate and classify (as essential, removable, or poles of some order) the singularities of

$$f(z) = \frac{e^{\pi/z}}{(z-\pi)^2}.$$

   Explain why each singularity is of the kind you state. Calculate the residues at the poles.

   The function is analytic whenever $z \neq \pi$ and $z \neq 0$, so these are the only possible singularities. $z = \pi$ is a pole of order 2 since we have $f(z) = g(z)/(z-\pi)^2$ for $g(z) = e^{\pi/z}$ analytic at $z = \pi$ with $g(\pi) \neq 0$. The residue at this pole is given by

$$\text{Res}(f,\pi) = \lim_{z\to\pi} \frac{d}{dz}(z-\pi)^2 f(z) = \lim_{z\to\pi} \frac{d}{dz} e^{\pi/z} = (-\pi/z^2)e^{\pi/z}|_{z=\pi} = \frac{-e}{\pi}.$$

   $z = 0$ is an essential singularity. To see this, observe that by the Casorati-Weierstrass theorem $e^{\pi/z}$ comes arbitrarily close to both 0 and every integer n in every punctured neighborhood of 0. Since the denominator $1/(z-\pi)^2$ has magnitude between $2/\pi^2$ and $1/2\pi^2$ in every sufficiently small punctured neighborhood (because it is continuous at $z = 0$), we conclude that the magnitude of $f(z)$ can be arbitrarily large as well as arbitrarily small in every punctured neighborhood of zero. Thus, it cannot be a removable singularity or a pole.

4. (7 points) Evaluate the integral

$$\oint_{|z|=1} z^4 e^{2/z^2}\, dz,$$

   oriented positively.

   The integrand has one singularity in the interior of the contour at $z = 0$. Since this is an essential singularity, we compute the first few terms of hte Laurent series:

$$z^4 \left(1 + (2/z^2) + \frac{(4/z^4)}{2!} + \frac{(8/z^6)}{3!} + \ldots\right).$$

   Since the coefficient of $1/z$ in this series is zero, the residue at 0 is zero, and the integral is also zero.

5. (8 points) Determine the number of zeros (counting multiplicity) of

$$f(z) = 2(z-1)^3 - e^{-z}$$

inside the open disk $D(1,1) = \{z : |z-1| < 1\}$.

We use Rouche's theorem with $f(z) = 2(z-1)^3$ and $g(z) = -e^{-z}$. Notice that both functions are analytic on and inside the contour C along $|z-1| = 1$ oriented positively, and that $f(z) \neq 0$ on C. Moreover, we have $|f(z)| = 2$ on C and $g(z) = |e^{-z}| = e^{-|\mathrm{Re}(z)|} \leq e^0 = 1$ on C, so $|f(z)| > |g(z)|$ on C. Thus, the function we are interested in, $f(z) + g(z)$, has the same number of zeros as $f(z)$ in the interior of C, which is just the open disk we are interested in. This number of zeros is three (since $f(z)$ has a zero of order 3 at $z = 1$).