Math 185-5 Fall 2015, Midterm 1 Solutions, N. Srivastava

1.

$$\left|\frac{2z-i}{2+iz}\right| = 1 \iff |2z-i|^2 = |2+iz|^2$$
$$\iff (2z-i)\overline{(2z-i)} = (2+iz)\overline{(2+iz)}$$
$$\iff (2z-i)(2\bar{z}+i) = (2+iz)(2-i\bar{z})$$
$$\iff 4|z|^2 + 2iz - 2i\bar{z} + 1 = 4 - 2i\bar{z} + 2iz + |z|^2$$
$$\iff 4|z|^2 + 1 = 4 + |z|^2,$$

which is true when $|z| = 1$.

2. $i^i = \exp(i\log(i)) = \exp(i(\ln(1) + i(\pi/2 + 2n\pi))) = \exp(-\pi/2 + 2n\pi)$ for $n \in \mathbb{Z}$. These are real numbers, so
$$(i^i)^{1/2} = \pm\exp(-\pi/4 + n\pi) \quad n \in \mathbb{Z}.$$

More systematically, one can see this by considering the polar forms
$$\exp(-\pi/2 + 2n\pi)e^{i(0+2\pi k)}, \quad n, k \in \mathbb{Z},$$
which have square roots
$$\exp(-\pi/4 + n\pi), \exp(-\pi/4 + n\pi)e^{i\pi}.$$

3. (a) FALSE. The interior of the simple closed contour $|z| = 2$ is not contained in the region.

(b) TRUE. The branch
$$\mathrm{Log}_0(z) = \ln|z| + i\mathrm{Arg}_0(z) \quad (0 < \mathrm{Arg}(z) < 2\pi)$$
is analytic on $\mathbb{C} \setminus [0, \infty)$, so $\mathrm{Log}_0(z-3)$ is analytic on $\mathbb{C} \setminus [3, \infty)$, which contains $D(0, 1)$.

(c) TRUE. $\lim_{z\to z_0} f(z) = L \neq 0$ and $\lim_{z\to z_0} 1 = 1$, so the limit of the quotient is
$$\lim_{z\to z_0} \frac{1}{f(z)} = \frac{1}{L}.$$

(d) FALSE. For instance $f(z) = z^2$ is entire, but $g(z) = \mathrm{Re}(f(z)) = x^2 - y^2$ for $z = x + iy$ is differentiable only at zero.

(e) FALSE. $f(z) = 1/z$ has y antiderivative $\mathrm{Log}(z)$ on $D_1 = \mathbb{C} \setminus (-\infty, 0]$ and antiderivative $\mathrm{Log}_0(z)$ on $D_2 = \mathbb{C} \setminus [0, \infty)$. But it cannot have an antiderivative on $D_1 \cup D_2 = \mathbb{C} \setminus \{0\}$ since
$$\oint_{|z|=1} \frac{1}{z} dz = 2\pi i \neq 0,$$
where the integral is taken positively.

4. (a) We have $f(x+iy) = u + iv$ for $u(x,y) = \sqrt{x^2+y^2}$ and $v(x,y) = 0$. When $x^2+y^2 \neq 0$ $u(x,y)$ is differentiable in x and y with partial derivatives

$$u_x = \frac{x}{\sqrt{x^2+y^2}} \quad u_y = \frac{y}{\sqrt{x^2+y^2}}.$$

Thus if either $x \neq 0$ or $y \neq 0$ the Cauchy-Riemann equations

$$u_x = v_y = 0 \quad u_y = -v_x = 0$$

cannot be satisfied, so f is not differentiable at any $z \neq 0$.

At $z = 0$ we observe that the limit of the difference quotient along the positive real axis:

$$\lim_{\Delta x \to 0^+} \frac{|\Delta x| - 0}{\Delta x} = 1$$

is not equal to the limit along the negative real axis

$$\lim_{\Delta x \to 0^-} \frac{|\Delta x| - 0}{\Delta x} = -1$$

so f is not differentiable at zero either.

(b) Assume for the sake of contradiction that

$$f(z) = \frac{z}{1+|z|}$$

is differentiable at $z \neq 0$. Since $1/z$ is also differentiable whenever $z \neq 0$, the product

$$g(z) = \frac{z}{1+|z|}\frac{1}{z} = \frac{1}{1+|z|}$$

must be differentiable at z. Since $g(z) \neq 0$ everywhere, its reciprocal

$$h(z) = \frac{1}{g(z)} = 1 + |z|$$

must also be differentiable at z. But this means that $h(z) - 1 = |z|$ is differentiable at z, which we showed is not true.

Thus, $f(z)$ is not analytic at any $z \neq 0$. It is not analytic at $z = 0$ either since every neighborhood of 0 contains a nonzero point, at which f cannot be differentiable.

5. Observe that

$$|\cos(\alpha + it)| = \frac{|e^{i\alpha + i^2 t} + e^{-i\alpha - i^2 t}|}{2} \leq \frac{|e^{i\alpha}|e^{-t} + |e^{-i\alpha}|e^{t}}{2} = \frac{e^{-t}+e^{t}}{2},$$

since α is real.

By the triangle inequality

$$\left|\int_0^1 \cos(\alpha + it)dt\right| \leq \int_0^1 |\cos(\alpha+it)|dt \leq 1 \cdot \max_{t \in [0,1]} \frac{e^{-t}+e^{t}}{2} \leq \frac{e^0 + e^1}{2} < 2,$$

as desired.

6. Write

$$\oint_C (\overline{z} + e^{iz})dz = \oint_C \overline{z}dz + \oint_C e^{iz}dz.$$

Since $\frac{d}{dz}\frac{e^{iz}}{i} = e^{iz}$ the second integrand has an antiderivative, and since C is closed the second integral must be zero.

For the first integral, we parameterize C as $z(t) = 2 + e^{it}, t \in [0, 2\pi]$ and write

$$\begin{aligned}\oint_C \overline{z}dz = \int_0^{2\pi} \overline{(2 + e^{it})}z'(t)dt &= \int_0^{2\pi} (2 + e^{-it})ie^{it}dt \\ &= 2i\int_0^{2\pi} e^{it}dt + i\int_0^{2\pi} dt \\ &= 2i\frac{e^{i2\pi} - e^{i0}}{i} + 2\pi i = 2\pi i.\end{aligned}$$