Math 185 Fall 2015, Sample Midterm 1 Solutions

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1. Writing $z = re^{i\theta}$ and $\overline{z} = re^{-i\theta}$, we have

$$r^{10}e^{i10\theta} = r \cdot r^{10}e^{-i10\theta},$$

so

$$r^{10} = r^{11} \quad \Rightarrow \quad r = 0 \quad or \quad 1.$$

In the first case $z = 0$. Otherwise, $e^{i10\theta} = e^{-i10\theta}$, so $e^{i20\theta} = 1$ and

$$20\theta = 2n\pi \quad \Rightarrow \quad \theta = \frac{n\pi}{10}, \quad n \in \mathbb{Z}.$$

Thus, $z = 0$ or $e^{in\pi/10}$, $n = 0, \ldots, 19$.

2. (15 points) True or false, (no need for justification):

(a) If f is continuous on a domain D, then f must be differentiable at at least one point in D.

False. Consider the continuous function $f(z) = \overline{z}$, which is differentiable nowhere on the domain $D = \{z : |z - 2| = 1\}$.

(b) If D_1 is a domain, D_2 is a domain, and $D_1 \cap D_2 \neq \varnothing$, then $D_1 \cup D_2$ is a domain.

True. $D_1 \cap D_2$ is open since an intersection of open sets is open. Each of D_1 and D_2 is connected. To see that the union is connected, let $z_0 \in D_1 \cap D_2$ and let $z_1 \in D_1$ and $z_2 \in D_2$ be arbitrary points. Since D_1 is connected there is a polygonal path P_1 from z_1 to z_0 contained in D_1, and since D_2 is connected there is a polygonal path P_2 from z_2 to z_0 contained in D_2. But now the concatenation of P_1 and P_2 is a polyogonal path from z_1 and z_2 contained in $D_1 \cup D_2$.

(c) The function

$$f(z) = \begin{cases} \frac{1}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is entire.

False. $f(z)$ is not continuous at $z = 0$ since

$$\lim_{z \to 0} \frac{1}{z} = \infty \neq f(0).$$

Thus f is not differentiable at $z = 0$.

(d) $\overline{\exp(\log(\overline{z}))} = z$ for all z.

True. From the definition of log, we have $\exp(\log(\overline{z})) = \overline{z}$. Taking conjugates of both sides gives the desired result.

(e) If f and f' are analytic on D and $f''(z) = 0$ for all $z \in D$ then f must be constant on D.

False. Consider the function $f(z) = z$, which is entire with $f''(z) = 0$ but not constant.

3. Find all points of differentiability of the function $f(z) = z + |z|^2$ defined on $\mathbb{C}$, and explain why f is differentiable at those points.

Write

$$f(x + iy) = x + iy + (x^2 + y^2) = x + x^2 + y^2 + iy = u + iv$$

for $u = x + x^2 + y^2$ and $v = y$. Note that u and v are differentiable everywhere with partial derivatives

$$u_x = 2x + 1 \quad u_y = 2y \quad v_x = 0 \quad v_y = 1,$$

which are continuous everywhere as they are polynomials. Thus, f is differentiable at all $z = x + iy$ where these derivatives satisfy the Cauchy-Riemann equations:

$$u_x = v_y \quad \Rightarrow \quad 2x + 1 = 1 \Rightarrow x = 0,$$

$$u_y = -v_x \quad \Rightarrow \quad 2y = 0 \Rightarrow y = 0.$$

In particular, $f(z)$ is differentiable at $z = 0$ and nowhere else.

4. Suppose f is analytic on a domain D, $|f(z)| \leq 1$, and $e^{f(z)} \in \mathbb{R}$ for all $z \in D$. Show that f must be constant on D.

Let $f(z) = u(z) + iv(z)$ for $u(z) = Re(f(z)), v = Im(f(z))$. Since $|f(z)| \leq 1$ for all $z \in D$ we must have $|u(z)| \leq 1$ and $|v(z)| \leq 1$ for all $z \in D$. Since $e^{f(z)} = e^{u(z)}e^{iv(z)} \in \mathbf{R}$, we know $v(z)$ must be an integer multiple of π. But $|v(z)| \leq 1$ so we must have $v(z) = 0$ for all $z \in D$. Now $f(z)$ is real for all $z \in D$, which by a result in class means that it must be constant.

5. Evaluate the integral:

$$\oint_C \frac{1}{\overline{z}^n} dz$$

where n is an integer and C is a circle of radius π centered at the origin, oriented positively. For which n does $f(z) = \frac{1}{\overline{z}^n}$ have an antiderivative in $\mathbb{C} \setminus \{0\}$?

Parameterizing the circle as $z(t) = e^{it}, t \in [0, 2\pi]$, we have

$$\oint_C \frac{1}{\overline{z}^n} dz = \int_0^{2\pi} \frac{ie^{it}}{e^{-itn}} dt = i \int_0^{2\pi} e^{it(1-n)} dt = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases},$$

by a calculation done in class.

The above shows that $f(z)$ does not have an antiderivative for $n = 1$ (since otherwise the integral along every closed contour would be zero). For $n = 0$ we have $f(z) = 1$, which does have an antiderivative (equal to z). For the remaining n, it turns out that $f(z)$ does not have an antiderivative either, and to show this we exhibit a (different) closed contour for which the integral is nonzero. See the handwritten PDF attachment for the calculation.

In retrospect, the last part of this problem was too long/difficult for the exam given the techniques we have learned so far. There will not be questions of this intricacy on the midterm. However, it is easy to solve once you know that the derivative of an analytic function must be analytic (which we will learn next week). Given this fact, here is a short proof that $1/\overline{z}^n$ cannot have an antiderivative for $n \neq 0$ or 1.

Assume it does, i.e, $F'(z) = 1/\overline{z}^n$ on $D = \mathbb{C} \setminus \{0\}$ for some analytic F. Recall that $\frac{d}{dz}\frac{1}{(-n+1)z^{n-1}} = \frac{1}{z^n}$ on D for $n \neq 1$, so the function

$$g(z) = F(z) + \frac{1}{(-n+1)z^{n-1}}$$

has derivative equal to

$$g'(z) = f(z) + \frac{1}{z^n} = \frac{z^n + \overline{z}^n}{z^n \overline{z^n}} = \frac{2Re(z^n)}{|z|^{2n}}.$$

But $g(z)$ must be analytic on D as it is the sum of two analytic functions, so $g'(z)$ is analytic. As it is real, it must be constant, which is only possible when $n = 0$.