Problem 5 on Sample Midterm

Let $n \neq 0$.

Assume $f(z) = \dfrac{1}{\bar{z}^n}$ has an antiderivative $F(z)$ on $D = \mathbb{C} \setminus \{0\}$.

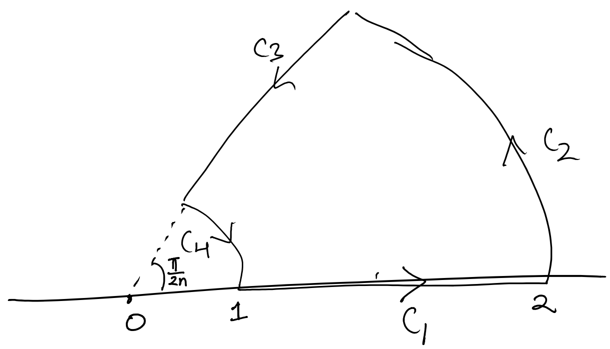
Since $\dfrac{1}{z^n}$ also has an antiderivative, this means the sum

$$h(z) = \frac{1}{\bar{z}^n} + \frac{1}{z^n} = \frac{2\,\mathrm{Re}\,(z^n)}{|z|^{2n}}$$ must also have

an antiderivative. We will show that this is impossible by producing a closed contour C_n (depending on n) for which

$$\oint_{C_n} h(z)\,dz \neq 0 .$$

Consider the contour $C_n = C_1 + C_2 + C_3 + C_4$



C_1: $\gamma_1(t) = t, \quad t \in [1,2]$

C_2: $\gamma_2(t) = 2e^{it} \quad t \in [0, \frac{\pi}{2n}]$

C_3: $\gamma_3(t) = (2-t)e^{i\frac{\pi}{2n}} \quad t \in [0,1]$

C_4: $\gamma_4(t) = e^{\frac{\pi}{2n} - t} \quad t = [0, \frac{\pi}{2n}]$.

Notice that ① $\displaystyle\int_{C_1} h(z)\,dz = \int_1^2 h(t)\,dt$ is real

② When $z \in C_3$ $\arg(z) = \frac{\pi}{2n}$ so

$$h(z) = \frac{2\,\mathrm{Re}\,\left(|z|^n e^{\frac{i\pi}{2n} \cdot n}\right)}{|z|^{2n}} = 2|z|^{-n}\,\mathrm{Re}(i) = 0$$

Thus, the imaginary part of $\oint_C h(z)\,dz$ is equal to

$$\mathrm{Im}\left(\int_{C_2} h(z)dz\right) + \mathrm{Im}\left(\int_{C_4} h(z)dz\right)$$

We now calculate: $$\int_{C_2} h(z)dz = \int_0^{\pi/2n} \frac{2\,\mathrm{Re}((2e^{it})^n)}{2^{2n}} ie^{it}\,dt$$

$$= \frac{2}{2^n} i \int \cos(nt)\left(\cos(t) + i\sin(t)\right) dt$$

So $$\mathrm{Im}\left(\int_{C_2} h(z)dz\right) = \frac{2}{2^n}\int_0^{\pi/2n} \cos(nt)\cos(t)\,dt$$

By a similar calculation $$\mathrm{Im}\left(\int_{C_4} h(z)dz\right) = -2\int_0^{\pi/2n} \cos(nt)\cos(t)\,dt$$

As $\int_0^{\pi/2n} \cos(nt)\cos(t) \neq 0$ these cannot be equal,

So $\mathrm{Im}\left(\int_C h(z)dz\right) \neq 0.$