Math 185-5 Fall 2015, Homework 9

Do not turn in, solutions will be posted on October 28

1. For book problems, see the handwritten pages.

2. Consider the function $g(z) = f(z) - w_0$, and note that $g(z)$ is analytic and nonzero on C. Since $g'(z) = f'(z)$ we have by the argument principle,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z) - w_0}dz = Z,$$

where Z is the number of zeros of $g(z)$ inside C, which is just the number of solutions of $f(z) = w_0$.

Though this was not asked in the question, observe that this integral is also equal to the winding number of the contour $g(C)$ (i.e., the contour with parameterization as $g(z(t))$ if C is parameterized as $z(t)$) around zero. This is the same as the winding number of the translated contour $g(C) + w_0 = f(C) - w_0 + w_0 = f(C)$ around w_0.

The point of this question is to say that there is nothing particularly special about zero, and other values w_0 can be handled simply by translating the function.

3. *This solution was rewritten on 12/7. The original skipped a couple of steps, and I think this is clearer.*

Let $\mathrm{Int}(C)$ denote the interior of a simple closed contour and ∂D denote the boundary of a region in the plane.

The key observation is that $f(C)$ cannot intersect itself, since if it did there would be two points $z_1, z_2 \in C$ such that $f(z_1) = f(z_2)$, contradicting that f is 1-1 on C. This means that $f(C)$ is a simple closed contour, so by the Jordan curve theorem it has an interior and an exterior, separated by a boundary $f(C)$[1]

By the open mapping theorem, this implies that if $z \in \mathrm{Int}(C)$ then $f(z) \notin f(C)$, i.e., every interior point of C gets mapped to an interior point of $f(C)$.

We will now show that f is $1 - 1$ in $\mathrm{Int}(C)$. Let $z_0 \in \mathrm{Int}(C)$ and $w_0 = f(z_0) \in \mathrm{Int}(f(C))$. Note that $w_0 \notin f(C)$, so the function

$$g(z) := f(z) - w_0$$

is nonzero on C, and the image contour $g(C)$ does not pass through zero. We will show that $f(z) - w_0$ has exactly one zero in $\mathrm{Int}(C)$, establishing that f is 1-1 there.

Recall that the winding number of a closed contour K around zero can be defined[2] as an integral

$$\mathrm{Wind}_0(K) = \frac{1}{2\pi i}\oint_K \frac{1}{z}dz.$$

[1] It is actually somewhat tricky to show that the boundary is equal to $f(C)$. Here is one argument. Let $S = f(C \cup \mathrm{Int}(C)) = f(C) \cup f(\mathrm{Int}(C))$ be the image of $C \cup \mathrm{Int}(C)$. Since f is continuous and $C \cup \mathrm{Int}(C)$ is closed, the image S is also closed, and $\partial S \subset S$. The open mapping theorem implies that $f(\mathrm{Int}(C)) \cap \partial S = \emptyset$, since $f(\mathrm{Int}(C))$ is an open set ∂S is a set of boundary points. Thus, we must have $\partial S \subseteq f(C)$. Since both $f(C)$ and ∂S are simple closed curves, this implies that $\partial S = f(C)$.

[2] As we saw in class, by writing the parameterization of K this is equivalent to counting the number of times K winds around zero. The latter quantity can be calculated by drawing any ray r emanating from zero and subtracting the number of times K crosses r clockwise from the number of times it crosses r counterclockwise.

In our case, we are interested in

$$\mathrm{Wind}_0(g(C)) = \frac{1}{2\pi i}\oint_{g(C)} \frac{1}{w}dw = \frac{1}{2\pi i}\oint_C \frac{g'(z)}{g(z)}dz.$$

If 0 is not in the interior then by Cauchy-Goursat this integral, and therefore the winding number, is zero. If it is in the interior then by Cauchy's integral formula it is one. In either case, since g is analytic on and inside C, the argument principle tells us that the number of zeros is equal to the winding number, so that there is at most one zero of $g(z)$ inside C. This means there is exactly one solution of $f(z) - w_0 = 0$ inside C, so f must be 1-1.

4. The simplest example is just $f(z) = z$ and $g(z) = -z$. $f(z)$ is nonzero on the unit circle and has one zero in its interior, but $f(z) + g(z) = 0$ everywhere.

   A more interesting example is $f(z) = e^{iz}$ and $g(z) = -e^{-iz}$ on the unit circle. $f(z)$ has no zeros in the interior, but the sum is $2i\sin(z)$, which has one zero.