1. For book problems, see the handwritten pages.

2. Consider the function $g(z) = f(z) - w_0$, and note that $g(z)$ is analytic and nonzero on $C$. Since $g'(z) = f'(z)$ we have by the argument principle,

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z) - w_0} dz = Z,$$

where $Z$ is the number of zeros of $g(z)$ inside $C$, which is just the number of solutions of $f(z) = w_0$.

Though this was not asked in the question, observe that this integral is also equal to the winding number of the contour $g(C)$ (i.e., the contour with parameterization as $g(z(t))$ if $C$ is parameterized as $z(t)$) around zero. This is the same as the winding number of the translated contour $g(C) + w_0 = f(C) - w_0 + w_0 = f(C)$ around $w_0$.

The point of this question is to say that there is nothing particularly special about zero, and other values $w_0$ can be handled simply by translating the function.

3. This solution was rewritten on 12/7. The original skipped a couple of steps, and I think this is clearer.

Let $\text{Int}(C)$ denote the interior of a simple closed contour and $\partial D$ denote the boundary of a region in the plane.

The key observation is that $f(C)$ cannot intersect itself, since if it did there would be two points $z_1, z_2 \in C$ such that $f(z_1) = f(z_2)$, contradicting that $f$ is 1-1 on $C$. This means that $f(C)$ is a simple closed contour, so by the Jordan curve theorem it has an interior and an exterior, separated by a boundary $f(C)^1$

By the open mapping theorem, this implies that if $z \in \text{Int}(C)$ then $f(z) \notin f(C)$, i.e., every interior point of $C$ gets mapped to an interior point of $f(C)$.

We will now show that $f$ is 1-1 in $\text{Int}(C)$. Let $z_0 \in \text{Int}(C)$ and $w_0 = f(z_0) \in \text{Int}(f(C))$. Note that $w_0 \notin f(C)$, so the function

$$g(z) := f(z) - w_0$$

is nonzero on $C$, and the image contour $g(C)$ does not pass through zero. We will show that $f(z) - w_0$ has exactly one zero in $\text{Int}(C)$, establishing that $f$ is 1-1 there.

Recall that the winding number of a closed contour $K$ around zero can be defined\(^2\) as an integral

$$\text{Wind}_0(K) = \frac{1}{2\pi i} \oint_K \frac{1}{z} dz.$$

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\(^1\)It is actually somewhat tricky to show that the boundary is equal to $f(C)$. Here is one argument. Let $S = f(C \cup \text{Int}(C)) = f(C) \cup f(\text{Int}(C))$ be the image of $C \cup \text{Int}(C)$. Since $f$ is continuous and $C \cup \text{Int}(C)$ is closed, the image $S$ is also closed, and $\partial S \subset S$. The open mapping theorem implies that $f(\text{Int}(C)) \cap \partial S = \emptyset$, since $f(\text{Int}(C))$ is an open set $\partial S$ is a set of boundary points. Thus, we must have $\partial S \subset f(C)$. Since both $f(C)$ and $\partial S$ are simple closed curves, this implies that $\partial S = f(C)$.

\(^2\)As we saw in class, by writing the parameterization of $K$ this is equivalent to counting the number of times $K$ winds around zero. The latter quantity can be calculated by drawing any ray $r$ emanating from zero and subtracting the number of times $K$ crosses $r$ clockwise from the number of times it crosses $r$ counterclockwise.
In our case, we are interested in

\[
\text{Wind}_0(g(C)) = \frac{1}{2\pi i} \oint_{g(C)} \frac{1}{w} dw = \frac{1}{2\pi i} \oint_C \frac{g'(z)}{g(z)} dz.
\]

If 0 is not in the interior then by Cauchy-Goursat this integral, and therefore the winding number, is zero. If it is in the interior then by Cauchy’s integral formula it is one. In either case, since \(g\) is analytic on and inside \(C\), the argument principle tells us that the number of zeros is equal to the winding number, so that there is at most one zero of \(g(z)\) inside \(C\). This means there is exactly one solution of \(f(z) - w_0 = 0\) inside \(C\), so \(f\) must be 1-1.

4. The simplest example is just \(f(z) = z\) and \(g(z) = -z\). \(f(z)\) is nonzero on the unit circle and has one zero in its interior, but \(f(z) + g(z) = 0\) everywhere.

A more interesting example is \(f(z) = e^{iz}\) and \(g(z) = -e^{-iz}\) on the unit circle. \(f(z)\) has no zeros in the interior, but the sum is \(2i\sin(z)\), which has one zero.