Math 185-05 Hw9 Selected book Problems

77.2

(a) The only singularity is a pole of order 2 at zero. The Laurent series at zero is:

$$\frac{1}{z^2}\left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} \cdots\right) = \frac{1}{z^2} - \frac{1}{z} + \frac{z}{2!} \cdots$$

So $\operatorname{Res}(0) = -1$ and the integral is $-2\pi i$.

(b) Here we have a pole of order 2 at $z = 1$ since $\exp(-1) \neq 0$. We may extract the residue by computing

$$\operatorname{Res}(-1) = \lim_{z \to 1} \frac{d}{dz} \frac{\cancel{(z-1)^2}\, e^{-z}}{\cancel{(z-1)^2}} = -e^{-z}\Big|_1 = -\frac{1}{e},$$

So the integral is $-\frac{2\pi i}{e}$.

(d) There are two singularities inside the contour — $z=0$ and $z=2$ — and by writing the integrand as $f(z) = \frac{z+1}{(z-2)z}$ we see both are simple:

$$\operatorname{Res}(2) = \cancel{(z-2)} \frac{z+1}{\cancel{(z-2)}z}\Bigg|_{z=2} = \frac{3}{2}$$

$$\operatorname{Res}(0) = \cancel{z} \frac{(z+1)}{(z-2)\cancel{z}}\Bigg|_{z=2} = -\frac{1}{2}$$

$$\text{So} \int_{|z|=3} f(z)\,dz = 2\pi i\left(\frac{3}{2} - \frac{1}{2}\right) = \underline{\underline{2\pi i}}$$

81.1 (a) Write $\frac{z+1}{z^2+9} = \frac{z+1}{(z-3i)(z+3i)}$,

so the singularities are $z = \pm 3i$ and these are simple poles since they are simple zeros of the denominator and the numerator is nonzero.

$$\operatorname{Res}(3i) = \frac{\cancel{(z-3i)}(z+1)}{\cancel{(z-3i)}(z+3i)}\Bigg|_{z=3i} = \frac{1+3i}{6i}$$

$$= \frac{3-i}{\ldots}$$

$$\left.\frac{\quad}{(\cancel{z-3i})(z+3i)}\right|_{z=3i}$$

$$= \frac{3-i}{6}$$

and similarly $\mathrm{Res}(-3i) = \left.\dfrac{(\cancel{z+3i})(z+1)}{(z-3i)(\cancel{z+3i})}\right|_{z=-3i}$

$$= \frac{-3i+1}{-6i} = \frac{3+i}{6}$$

(81.2) (b) We have $\dfrac{\mathrm{Log}(z)}{(z^2+1)^2} = \dfrac{\mathrm{Log}(z)}{(z+i)^2(z-i)^2}$,

So there are poles of order 2 at $z = \pm i$ since $\mathrm{Log}(\pm i) \neq 0$.

$$\mathrm{Res}(i) = \frac{d}{dz}\left.\frac{\cancel{(z-i)^2}\,\mathrm{Log}(z)}{\cancel{(z-i)^2}(z+i)^2}\right|_{z=i}$$

$$= \left.\frac{d}{dz}\mathrm{Log}\,z \cdot \frac{1}{(z+i)^2} + \mathrm{Log}(z)\left(\frac{-2}{(z+i)^3}\right)\right|_{z=i}$$

$$= \frac{1}{z(z+i)^2} - \frac{\mathrm{Log}(z)}{(z+i)^3} = \frac{1}{i(-4)} - \frac{(i\pi/2)}{(-8i)}$$

$$= \frac{\pi}{16} + \frac{1}{4}i .$$

(81.6) The integrand $f(z) = \dfrac{\cosh \pi z}{z(z-i)(z+i)}$

has singularities at $z = 0, \pm i$, all of which are simple since $\cosh(0), \cosh(i\pi), \cosh(-i\pi) \neq 0$, and all are contained in C.

The corresponding residues are:

$$\mathrm{Res}(0) = \frac{\cosh(0)}{(0-i)(0+i)} = \frac{1}{-i^2} = 1$$

$$\mathrm{Res}(i) = \frac{\cosh(i\pi)}{i(2i)} = \frac{\cos\pi}{-2} = -\frac{\cos\pi}{2} = 1/2$$

$$\mathrm{Res}(-i) = \frac{\cosh(-i\pi)}{(-i)(-2i)} = \frac{\cos(-\pi)}{2} = 1/2$$

So the integral is $2\pi i(1 + 1/2 + 1/2) = \underline{4\pi i}$

(83.5a) $\tan z = \dfrac{\sin z}{\cos z}$ has poles at $n\pi + \pi/2$, the zeros of $\cos z$.

The only two contained in $|z| = 2$ are $\pm\frac{\pi}{2}$, and these are simple

since $\cos(z) = -\sin(z-\pi/2) = -(z-\pi/2) + (z-\pi/2)^3$

The only two contained in $|z|=2$ are $\pm\frac{\pi}{2}$, and ...

since $\cos(z) = -\sin(z-\pi/2) = -(z-\pi/2) + \frac{(z-\pi/2)^3}{3!} \ldots$

has a simple zero at $z=\pi/2$ and similarly at $z=-\pi/2$.

The residues are thus:

$$\mathrm{Res}(\pi/2) = \lim_{z\to\pi/2} \frac{(z-\pi/2)\sin(z)}{-\sin(z-\pi/2)} = -\sin(\pi/2)\cdot\lim_{z\to\pi/2}\frac{z-\pi/2}{\sin(z-\pi/2)}$$

$$= -1$$

$$\mathrm{Res}(-\pi/2) = \lim_{z\to-\pi/2} \frac{(z+\pi/2)\sin(z)}{\sin(z+\pi/2)} = \sin(-\pi/2)\cdot\lim_{z\to-\pi/2}\frac{z+\pi/2}{\sin(z+\pi/2)}$$

$$= -1$$

and a zero of order 7 at $z=1/2$,

So $\Delta_C \arg(f(z)) = 2\pi(7-3) = \underline{\underline{8\pi}}$.

94.6

(a) Let $f(z) = -5z^4$ and $g(z) = z^6 + z^3 - 2z$

and notice that $|g(z)| \leq |z|^6 + |z|^3 + 2|z| \leq 4 < 5 = |f(z)|$

wherever $|z|=1$. Moreover $f(z)\neq 0$ on C and is analytic everywhere.

Thus, by Rouché's theorem, $(f+g)(z)$ has the same number of zeros as $f(z)$ inside C, which is $\underline{\underline{4}}$.

(b) Take $f(z)=9$ and $g(z) = 2z^4 - 2z^3 + 2z^2 - 2z$

and notice $|g(z)| \leq 8 < |f(z)|$ on C.

Thus $(f+g)(z)$ has no zeros inside C.

94.8

Applying Rouché to the contour $|z|=2$ with $f(z) = 2z^5$ and $g(z) = -6z^2 + z + 1$, we find that the equation has $\underline{\underline{5}}$ zeros inside $|z|=2$.

Applying Rouché to $|z|=1$ with $f(z)=-6z^2$ and $g(z)=2z^5+z+1$,

we see that it has $\underline{\underline{2}}$ zeros inside $|z|=1$.

Thus, the number of zeros in the annulus $1<|z|<2$ is $5-2=\underline{\underline{3}}$.