77.2
(a) The only singularity is a pole of order 2 at 20.

The Laurent series at 20 is:
\[
\frac{1}{z^2}(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} \ldots) = \frac{1}{2^2} - \frac{1}{2} + \frac{z}{2^3} \ldots
\]

So \(\text{Res}(0) = -1\) and the integral is \(-2\pi i\).

(b) Here we have a pole of order 2 at \(z = 1\) since \(\exp(-1) \neq 0\). We may extract the residue by computing

\[
\text{Res}(-1) = \lim_{z \to 1} \frac{(z-1)^2 e^{-z}}{(z-1)^2} = -e^{-1} = -\frac{1}{e},
\]

so the integral is \(-\frac{2\pi i}{e}\).

(c) There are two singularities inside the contour—\(z = 0\) and \(z = 2\)—and by writing the integrand as \(\frac{z+1}{(z-2)^2}\) we see both are simple:

\[
\text{Res}(2) = \frac{3}{2}, \quad \text{Res}(1) = \frac{3}{2}
\]

So \(\int_{\gamma} f(z) \, dz = 2\pi i \left( \frac{3}{2} - \frac{1}{2} \right) = \pi i\).

81.1
(a) While \(\frac{z+1}{z^2+9} = \frac{z+1}{(z-3i)(z+3i)}\) \(f\)

So the singularities are \(z = \pm 3i\) and there are simple poles since they are simple zeros of the denominator and the numerator is nonzero.

\[
\text{Res}(3i) = \frac{(z-3i)(z+1)}{(z-3i)(z+3i)} \bigg|_{z = -3i} = \frac{1}{6i} = \frac{1}{i}
\]

\[
= 3 - i
\]
\[
\text{and similarly, } \text{Res}(-3i) = \left. \frac{z^{-1}}{6i} \right|_{\text{at } z = -3i} = \frac{-3i + 1}{-6i} = \frac{3i + 1}{6i}.
\]

(8.4.3) (b) We have
\[
\frac{\text{Res}(\log(z))}{(z + 1)^2} = \frac{\text{Log}(z)}{(z + 1)^2(z - 1)^2},
\]
So we are poles of order 2 at \( z = \pm i \). Since \( \text{Log}(\pm i) \neq 0 \),
\[
\text{Res}(i) = \frac{d}{dz} \left. \frac{\text{Log}(z)}{(z + 1)^2} \right|_{z = i} = \frac{d}{dz} \left. \frac{1}{(z + 1)^2 + \text{Log}(z) \left( \frac{1}{z + 1} \right)} \right|_{z = i} = \frac{1}{z(z + 1)^2} - \frac{\text{Log}(z)}{(z + 1)^2} = \frac{1}{i(-4)} - \left( \frac{i \sqrt{2}}{-8i} \right) = \frac{1}{16} + \frac{1}{4}i.
\]

(8.4.6) The integrand \( f(z) = \frac{\cosh(z)}{z(z - i)(z + 1)} \) has singularities at \( z = 0, \pm i \), all of which are simple since \( \cosh(0), \cosh(i\pi), \cosh(-i\pi) \neq 0 \), and all are contained in \( C \). The corresponding residues are:
\[
\text{Res}(0) = \left. \frac{\cosh(z)}{z(z - i)(z + 1)} \right|_{z = 0} = \frac{1}{-1} = \frac{1}{i},
\]
\[
\text{Res}(i) = \left. \frac{\cosh(i\pi)}{i(z + 1)} \right|_{z = i} = \frac{\cos \frac{\pi}{2} - \cos \frac{3\pi}{2}}{2} = \frac{1}{2},
\]
\[
\text{Res}(-i) = \left. \frac{\cosh(-i\pi)}{z(-2i)} \right|_{z = -i} = \frac{\cos(-\frac{\pi}{2})}{2i} = \frac{1}{2i}.
\]

So the integral is \( 2\pi i \left( \frac{1}{2} + \frac{1}{2i} \right) = \frac{\pi}{i} \).

(8.3.50) \( \tan z = \frac{\sin z}{\cos z} \) has poles at \( n\pi + \frac{\pi}{2} \), the zeros of \( \cos z \).

The only two contained in \( 12\text{ by } 2 \) are \( \pm \frac{\pi}{2} \), and hence are simple.
\[
\sin z = \cos \left( \frac{\pi}{2} - z \right) = -\sin \left( \frac{\pi}{2} - z \right) = -\sin(\pi - z) + (e^{-\pi z})^3
\]
The only two contained in \( \{ z \mid |z| = 1 \} \) are \( \frac{\pi}{2} \) and \( -\frac{\pi}{2} \).

Since \( \cos(z) = -\sin(z - \frac{\pi}{2}) = -\left( z - \frac{\pi}{2} \right) + \frac{(z - \frac{\pi}{2})^3}{3!} \ldots \)

has a simple zero at \( z = \frac{\pi}{2} \) and similarly at \( z = -\frac{\pi}{2} \).

The residues are thus:

\[
\text{Res}(\pi/2) = \lim_{z \to \pi/2} \left( z - \frac{\pi}{2} \right) \frac{\sin(z)}{-\sin(z - \frac{\pi}{2})} = -\sin(\frac{\pi}{2}) \cdot \lim_{z \to \pi/2} \frac{z - \pi/2}{\sin(z - \pi/2)} = -1
\]

\[
\text{Res}(-\pi/2) = \lim_{z \to -\pi/2} \left( z + \frac{\pi}{2} \right) \frac{\sin(z)}{\sin(z + \pi/2)} = \sin(-\pi/2) \cdot \lim_{z \to -\pi/2} \frac{z + \pi/2}{\sin(z + \pi/2)} = -1/1
\]

and a zero of order 7 at \( z = \frac{\pi}{2} \).

So \( \Delta_{\text{arg}(\pi/2)} = 2\pi(7-3) = 8\pi \).

(94.6) (a) Let \( f(z) = 5z^4 \) and \( g(z) = z^6 + z^3 + 2z^2 \)

and notice that \( |g(z)| < 1|z|^4 + |z|^3 + |z|^2 \leq 4 \leq 5 = |f(z)| \)

for \( |z| = 1 \). Moreover \( f(z) \neq 0 \) on \( C \) and is analytic everywhere.

Thus, by Rouché's theorem, \( f(z) + g(z) \) has the same

number of zeros as \( f(z) \) inside \( C \), which is 4.

(b) Take \( f(z) = 9 \) and \( g(z) = 2z^4 - 2z^3 + 2z^2 + 2z \)

and notice \( |g(z)| \leq 8 < |f(z)| \) on \( C \).

Thus \( f(z) + g(z) \) has no zeros inside \( C \).

(94.8) Applying Rouché to the contour \( 1|z| = 2 \) with \( f(z) = 2z^5 \)

and \( g(z) = -6z^4 + 2z^2 + 1 \), we find that the equation

...
has $5 \text{ zeros inside } |z|=2$. Applying Rouche to $|z|=1$ with $f(z) = -6z^2$ and $g(z) = 2z^3 + 3z + 1$, we see that it has $8 \text{ zeros inside } |z|=1$. Thus, the number of zeros on the annulus $|z|=1 \leq 1$ is $5 - 2 = 3$. 