Math 185-5 Fall 2015, Homework 8 Selected Solutions

1. (a) Observe that the function $f(z) = \frac{1}{1+z^2}$ has poles at $\pm i$, so it cannot have a Taylor series which converges in an open disk of radius greater than 1. On the other hand, if the real series in the question converged in any $(-r, r)$ interval of radius $r > 1$, then it would converge absolutely, which means that the same power series in *complex* z would also converge for all $|z| < r$. This means it would define an analytic function in $D(0, r)$. But by the identity theorem, there is only one function analytic in $D(0, 1)$ which agrees with $f(z) = \frac{1}{1+z^2}$ for $z \in (-1, 1)$. By uniqueneness of Taylor series, this would give a Taylor series for $f(z)$ convergent in $D(0, r)$ with $r > 1$, a contradiction.

   (b) The function $f(z) = \exp(1/z^2)$ is not analytic (in fact, it has an essential singularity) at $z = 0$, so it cannot have a convergent Taylor series expansion in any neighborhood of $z = 0$.

4 An example of such a function whose zeros have a limit point at $z = 1$ is

$$\sin(1/(1-z))$$

, since it is zero at $z_n = 1 - 1/n\pi$. To get the other limit points, we simply rotate the function by multiples of $\pi/2$, in particular the function

$$f(z) = \sin(1/(1-z))\sin(1/(1+z))\sin(i/(i-z))\sin(i/(i+z))$$

is analytic in $D(0, 1)$ and has the required properties.

68.5* All three domains are centered at zero, so we will look for Laurent series about zero. We will construct these by combining geometric expansions in z (which converge in the open interior of a disk centered at zero) and geometric expansions in $1/z$ (which converge in the exterior of a disk centered at zero).

(a) The desired region is $|z| < 1$, the interior of an open disk, so we expect to have only series in z.

$$\frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-z/2} = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}(z/2)^n,$$

where the first series converges in $|z| <$ and the second in $|z| < 2$, so their sum converges in $|z| < 1$. Rearranging shows that this is the same as $\sum_{n=0}^{\infty}(2^{-n-1} - 1)z^n$.

(b) Here we expect to have a series in $1/z$ (corresponding to the exterior of a disk $|z| > 1$) and a series in $z/2$ (corresponding to the interior $|z| < 2$). Thus, we choose to expand as:

$$\frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-1/z} + \frac{1}{2}\frac{1}{1-z/2} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \frac{1}{2}\sum_{n=0}^{\infty}(z/2)^n,$$

where the region of convergence is the region common to both series, namely $1 < |z| < 2$.

(c) Here, we expect to have series in $1/z$ and $2/z$ only, so we expand:

$$\frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-1/z} - \frac{1}{z}\frac{1}{1-2/z} = \sum_{n=0}^{\infty}\frac{1}{z^n+1} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}.$$

The region of convergence of the first part is $|z| > 1$ and that of the second is $|z| > 2$, so the sum converges in $|z| > 2$, as desired.

*5ac (a) The singularities occur at the zeros of the denominator. Recall that the zeros of $\sin(z)$ are equal to $n\pi$ for integers n (or see section 38). Since $\sin(z)$ has a simple zero at each $n\pi$, which is seen for instance by evaluating the derivative

$$\lim_{z\to n\pi} \frac{\sin(z)-\sin(n\pi)}{z-n\pi} = \cos(n\pi) = \pm 1 \neq 0$$

, the denominator has a simple zero at $n\pi$ for $n \neq -1$ and a double zero for $n=-1$. Thus, the function has simple poles at $z = n\pi, n \neq -1$ and a double pole at $z=-\pi$.

(c) The function $f(z) = \sin(1-1/z)$ is analytic whenever $z \neq 0$ since it is a composition of functions analytic at all such z. When $z=0$, observe that along the sequence $z_n = 1/(1+n\pi)$ converging to zero, we have

$$f(z_n) = 0.$$

Thus, $\lim_{z\to 0}|f(z)| \neq \infty$ so 0 is not a pole. By considering the sequence $y_n = 1/(1+in\pi)$, also converging to zero, we find that

$$f(y_n) = \sin(in\pi) = \frac{e^{-n\pi}-e^{n\pi}}{2i}$$

is unbounded as $n \to \infty$, so zero cannot be a removable singularity. Thus, it must be an essential singularity.

There are other ways to see this — any method that shows that the limit $\lim_{z\to 0} f(z)$ does not exist (and is not infinity) will show that it is an essential singularity. For example, you could consider the sequence $x_n = 1/(1-n\pi-\pi/2)$ converging to zero, for which $f(x_n) = (-1)^n$, so the sequence $f(x_n)$ has no limit. This is only possible when 0 is an essential singularity.

6* (a) By problem 3 (or by long division) we have

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{z}{3!} + O(z^3)$$

Since we are only interested in the principal part, these first few terms will suffice (in particular, since the most negative power that appears is $1/z$, terms of degree 2 or greater cannot contribute to the principal part). Taking the square, we find

$$\frac{1}{\sin^2(z)} = \left(\frac{1}{z} + \frac{z}{3!} + O(z^3)\right)\left(\frac{1}{z} + \frac{z}{3!} + O(z^3)\right) = \frac{1}{z^2} + \frac{2}{3!} + O(z^2)$$

so the principal part is just $1/z^2$.

(b) Observe that

$$\frac{1}{e^z-1} = \frac{1}{z+z^2/2+O(z^3)} = \frac{1}{z}\frac{1}{1+z/2+O(z^2)} = \frac{1}{z}\left(1-z/2+O(z^2)\right).$$

Multiplying by e^z+1 we have

$$\frac{e^z+1}{e^z-1} = \frac{1}{z}\left(1-z/2+O(z^2)\right)\left(2+z+O(z^2)\right) = \frac{1}{z}\left(2+O(z^2)\right),$$

so the principal part is just $2/z$.

7 True or False.

(a) Suppose $f(z)$ is analytic in $D(0,1)$ and $\{z_n\}$ is a sequence of points in $D(0,1)$ with $\lim_{n\to\infty} z_n = 0$ and $f(z_n) = \sin(z_n)$ for all n. Then f must equal $\sin(z)$ in $D(0,1)$.

TRUE. Both $\sin(z)$ and f are analytic in $D(0,1)$ and agree on a sequence with a limit point in $D(0,1)$, so by the identity theorem they must be the same.

(b) There exists an $f(z)$ analytic in $D(0,1)$ and a sequence $z_n \to 0$ contained in $D(0,1)$ such that $f(z_n) = n$ for all n.

FALSE. This implies that $\lim_{n\to\infty} |f(z_n)| = \infty$, so f cannot be continuous (and therefore can't be analytic) at 0.

(c) There exists an $f(z)$ analytic in $D(0,1)$ and a sequence $z_n \to 0$ in $D(0,1)$ such that $f(z_n) = z_n$ for even n and $f(z_n) = 0$ for odd n.

The identically zero sequence trivially has this property, so I should have said nontrivial sequence. For any nontrivial sequence (i.e., $z_n \neq 0$) , this is FALSE. By considering the odd points at which $f(z_n) = 0$, the identity theorem says that the only function analytic in $D(0,1)$ with this property is the zero function. But then $f(z_n) \neq z_n$ at the even points.

(d) If f and g have a pole at z_0 then $f+g$ has a pole at z_0.

FALSE. You could have $f = -g$.

(e) If f has a pole at z_0 and g has an essential singularity at z_0 then $f+g$ has an essential singularity at z_0.

TRUE. Adding the Laurent series, the sum will still have infinitely many terms in the principal part (as there is no way to cancel them with the finitely many terms from f).

(f) If f has a pole of order m at z_0 and g has a zero of order $n \leq m$ at z_0 then $f \cdot g$ has a removable singularity at z_0.

FALSE. Take $n = 1$ and $m = 2$. Then fg has a pole of order 1 at z_0.

(g) If $f(z)$ has a pole of order m at z_0 then $f'(z)$ has a pole of order $m+1$ at z_0.

TRUE. Differentiate $f(z) = g(z)/(z-z_0)^m$.

10* Let us handle the case $k = 1$ first. Suppose f is analytic in D with a single zero z_1 of order m_1. Then the function $g_1(z) = \frac{f(z)}{(z-z_1)^{m_1}}$ is analytic in $D \setminus \{z_1\}$, and undefined (i.e., has an isolated singularity) at z_1. However, this singularity at z_1 is removable since $\lim_{z\to z_1} \frac{f(z)}{(z-z_1)^{m_1}}$ exists and is nonzero (because $f(z)$ has a zero of order m_1). Remove the singularity by defining $g_1(z)$ at z_1 as

$$g_1(z_1) = \lim_{z\to z_1} fracf(z)(z-z_1)^{m_1}.$$

Now $g_1(z)$ is analytic in D and has the required properties.

For the general case, applying the above argument yields a function $g_1(z)$ analytic in D with $f(z) = g_1(z)(z-z_1)^{m_1}$ and zeros $z_2, \ldots, z_k$ of orders $m_2, \ldots, m_k$. By induction we can factorize $g_1(z)$ as $g(z)(z-z_2)^{m_2} \ldots (z-z_k)^{m_k}$, so we are done.

One may also prove this directly by observing that:

$$g(z) = \frac{f(z)}{(z-z_1)^{m_1} \ldots (z-z_k)^{m_k}}$$

has removable singlularities at $z_1, \ldots, z_m$, since for each z_i one has $f(z) = h_i(z)(z-z_i)^{m_i}$ for analytic $h_i(z_i) \neq 0$ in a neighborhood of z_i, implying that the limit

$$\lim_{z\to z_i} \frac{f(z)}{(z-z_i)^{m_i}} \neq 0.$$

Redefining

$$g(z_i) = \lim_{z\to z_i} \frac{f(z)}{(z-z_1)^{m_1} \ldots (z-z_k)^{m_k}}$$

at these singularities removes them and yields a function analytic in D with the desired properties.

The point of this question was to emphasize that removable singularities can simply be removed by redefining the function at them to obtain true analytic functions. The reason, as discussed in class, is that the redefined function's Laurent series is actually a Taylor series, so it is analytic at the (removed) singularity. Moreover, the redefined function agrees with the original function at all points other than the singularity.

11* The Casorati-Weierstrass theorem (section 84) says that if a function f has an essential singularity at z_0, then for every $\epsilon > 0$, $\delta > 0$, and w, there is a point $z \in D^o(z_0, \delta)$ such that $|f(z) - w| < \epsilon$, i.e., f comes arbitrarily close to every point in the complex plane in every punctured neighborhood of z_0. We will use this fact to rule out the possiblities that $e^{f(z)}$ has a removable singularity or a pole at z_0. First observe that $g(z) = e^{f(z)}$ cannot be bounded in any punctured neighborhood of z_0, since it comes arbitrarily close to every integer (say) in every punctured neighborhood of z_0. Thus, g cannot have a removable singularity at z_0. On the other hand, it also comes arbitrarily close to zero in every punctured neighborhood, so $\lim_{z \to z_0} |g(z)| \neq \infty$. Thus $g(z)$ cannot have a pole. The only remaining possibility is an essential singularity.