Math 185-5 Fall 2015, Homework 8

Due October 23 in class

1. 65.11, 68.5, 68.7, 68.10, 72.6, 72.10.
2. Based on the properties of complex power series and the identity theorem, give an explanation for the following "pathological" facts in real analysis:
   (a) The series
   $$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 \ldots$$
   converges only in the interval $|x| < 1$ even though $f(x)$ does not "blow up" at $x = \pm 1'$.
   (b) The function
   $$f(x) = \exp(-1/x^2)$$
   is infinitely differentiable at $x = 0$, but its Taylor series does not converge in any open interval around 0.
3. Division of power series is often more conveniently handled by appropriately using asymptotic notation and known expansions, rather than by solving linear equations or performing long division. As an example, observe that
   $$\frac{1}{\sin z} = \frac{1}{z - z^3/3! + O(z^5)} = \frac{1}{z}\frac{1}{1 - z^2/3! + O(z^4)}.$$
   Since $z^2/3! + O(z^4)$ tends to 0 as $z \to 0$, we have in particular $|z^2/3! + O(z^4)| < 1$ for sufficiently small z (which is all we are interested in). Thus, the geometric expansion is valid in a neighborhood of 0, and we get
   $$\frac{1}{\sin(z)} = \frac{1}{z}\left(1 + (z^2/3! + O(z^4)) + O(z^4)\right) = \frac{1}{z} + \frac{z}{3!} + O(z^3),$$
   which is consistent with the result obtained by long division.

   Use the above method to find the first few terms of the Laurent series about 0 for $\frac{1}{e^z-1}$ and $\cot(z)$.
4. Give an example of a function which is analytic and not identically zero in $D(0,1)$, and whose zeros have limit points at $\{-1, 1, i, -i\}$.
5. Locate and classify (as removable, essential, or poles of a specified order) the singularities of the following functions:
   $$\frac{1}{(\pi+z)\sin(z)}, \quad \frac{1}{(z^2+z+1)^3}, \quad \sin\left(1 - \frac{1}{z}\right),$$
   $$\frac{1}{(2\cos z - 2 + z^2)^2}, \quad \operatorname{cosec}(\sin(z)).$$
6. Find the principal parts of the Laurent expansions about 0 of:
   $$\frac{1}{\sin^2(z)}, \quad \frac{e^z+1}{e^z-1}.$$

   Indicate the type of singularity in each case.

7. True or false (and explain why or give a counterexample):

   (a) Suppose $f(z)$ is analytic in $D(0,1)$ and $\{z_n\}$ is a sequence of points in $D(0,1)$ with $\lim_{n\to\infty} z_n = 0$ and $f(z_n) = \sin(z_n)$ for all n. Then f must equal $\sin(z)$ in $D(0,1)$.

   (b) There exists an $f(z)$ analytic in $D(0,1)$ and a sequence $z_n \to 0$ contained in $D(0,1)$ such that $f(z_n) = n$ for all n.

   (c) There exists an $f(z)$ analytic in $D(0,1)$ and a sequence $z_n \to 0$ in $D(0,1)$ such that $f(z_n) = z_n$ for even n and $f(z_n) = 0$ for odd n.

   (d) If f and g have a pole at z_0 then $f + g$ has a pole at z_0.

   (e) If f has a pole at z_0 and g has an essential singularity at z_0 then $f + g$ has an essential singularity at z_0.

   (f) If f has a pole of order m at z_0 and g has a zero of order $n \leq m$ at z_0 then $f \cdot g$ has a removable singularity at z_0.

   (g) If $f(z)$ has a pole of order m at z_0 then $f'(z)$ has a pole of order $m + 1$ at z_0.

8. Give examples of functions f_1, f_2, f_3 with the following properties:

   (a) f_1 has a simple pole at $z = 0$ and an essential singularity at $z = 1$.

   (b) f_2 has a removable singularity at $z = 0$, a pole of order 6 at $z = 1$, and an essential singularity $z = i$.

   (c) f_3 has non-isolated singlurities accumulating at ± 1 and a set of simple poles.

9. Suppose S is a finite subset of $D(0,1)$ and f is continuous and analytic in $D \setminus S$. Show that f must be analytic in $D(0,1)$.

10. Suppose f is analytic in a domain D and has zeros $z_1, \ldots, z_k \in D$ of orders $m_1, \ldots, m_k$. Prove that there is a function $g(z)$ analytic in D such that

$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \ldots (z - z_k)^{m_k} g(z).$$

    (hint: handle the case $k = 1$ first and use induction)

11. Show that if $f(z)$ has an essential singularity at z_0 then $e^{f(z)}$ also has an essential singulrity at z_0 (hint: consider the behavior of the limit as $z \to z_0$).

12. Suppose $f(z)$ is entire and there is a constant M such that

$$|f(z)| \leq M|\sin(z)| \quad \forall z \in \mathbb{C}.$$

    Show that $f(z) = K\sin(z)$ for some constant K. (hint: consider $f(z)/\sin(z)$.)