Math 185-5 Fall 2015, Homework 7

Due October 16 in class

73.1* Expanding $1/(z^2+1)$ as a geometric series in z^2, we can write the expression as

$$\frac{1}{z}\left(1+z+z^2/2!+z^3/3!\dots\right)\left(1-z^2+z^4+\dots\right)=\frac{1}{z}\left(1+z+(z^2/2!-z^2)+(-z^3+z^3/3!)+\dots\right)$$
$$=\frac{1}{z}+1-z/2-5z^2/6+\dots,$$

as desired.

5* Observe that the series converges in the open disk $|z|<3$ by the ratio test, since

$$\lim_{n\to\infty}\left|\frac{(n+1)^3z^{n+1}/3^{n+1}}{n^3z^n/3^n}\right|=\lim_{n\to\infty}|1+1/n|^3|z/3|<1,$$

whenever $|z|<3$. Thus, $f(z)$ is analytic in $D(0,3)$ and by uniqueness the above series is its Taylor series, with coefficients given by $f^{(n)}(0)/n!=n^3/3^n$. We then have

$$f^{(6)}(0)=6!\cdot 6^3/3^6.$$

For the second part, since f is analytic on and inside $|z|=1$, we apply Cauchy's integral formula for higher derivatives to get:

$$\oint_{|z|=1}\frac{f(z)}{(z-0)^4}dz=\frac{2\pi i}{3!}f^{(3)}(0)=\frac{2\pi i}{3!}\cdot 3!\cdot 3^3/3^3=2\pi i.$$

For the third part, we observe that $e^zf(z)$ is analytic on and inside the contour, so the integral is zero by Cauchy-Goursat. For the fourth part, Cauchy's integral formula reveals that the integral is

$$2\pi i(f(z)\sin(z))'(0)=2\pi i(f'(0)\sin(0)+f(0)\cos(0))=2\pi i(f'(0)\cdot 0+f(0))=2\pi i f(0)=0.$$

7* The principal branch $\mathrm{Log}(z)$ is analytic everywhere except the nonpositive real axis, so in particular it is analytic in an open disk of radius 2 centered at $z_0=2$. There are several ways to do this but perhaps the simplest is to observe that

$$\frac{d}{dz}\mathrm{Log}(z)=1/z$$

whenever $z\neq 0$ (this is true for every branch of the logarithm). Expanding $1/z$ as a power series about z_0, we have:

$$\frac{1}{z}=\frac{1}{2+(z-2)}=\frac{1}{2}\frac{1}{1+(z-2)/2}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{2^n},$$

whenever $|z-2|<2$. Now, if the power series for $\mathrm{Log}(z)$ is given by $\sum_{n=0}^{\infty}a_n(z-2)^n$, we must have by termwise differentiation:

$$\frac{d}{dz}\mathrm{Log}(z)=\sum_{n=0}^{\infty}\frac{d}{dz}a_n(z-2)^n=0+\sum_{n=1}^{\infty}na_n(z-2)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(z-2)^n.$$

But this must be equal to the series

$$\frac{1}{z} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(z-2)^n}{2^n}$$

which we computed above, so comparing the coefficients of $(z-2)^n$ on both sides, we find that

$$(n+1)a_{n+1} = \frac{(-1)^n}{2^{n+1}}, n \geq 0 \quad \Rightarrow \quad a_n = \frac{(-1)^{n-1}}{2^n n}, n \geq 1.$$

This only gives the coefficients $a_1, a_2, \ldots$; to find a_0 we simply evaluate at $z_0 = 2$ and see that $a_0 = \text{Log}(2) = \ln(2)$. Thus, the series is

$$\text{Log}(z) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}(z-2)^n.$$

The largest disk around z_0 in which $\text{Log}_{\pi/2}(z)$ is analytic is also $|z-2| < 2$. Since $\text{Log}_{\pi/2}(z) = \text{Log}(z)$ in this disk, we have the same Taylor expansion for $\text{Log}_{\pi/2}$ around 2.

8* Since f is entire, it has a Taylor expansion about zero:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

convergent everywhere, with coefficients $a_k = f^{(k)}(0)/k!$. By Cauchy's inequality (recall: this is obtained by applying the ML estimate to Cauchy's integral formula for higher derivatives), we can estimate the derivates as:

$$|f^{(k)}(0)| \leq \frac{k! M_R}{R^k},$$

where $M_R = \max_{z \in C_R} |f(z)|$, and C_R is a circle of radius R centered at zero. The question assumes that $M_R \leq M \cdot R^n$ for some fixed M, n, so for $k > n$ we have:

$$|f^{(k)}(0)| \leq \frac{k! M R^n}{R^k} = \frac{k! M}{R^{k-n}}.$$

Letting $R \to \infty$, we find that $|f^{(k)}| \leq \epsilon$ for every $\epsilon > 0$, which means $f^{(k)}(0) = 0$ for $k > n$. Thus, the Taylor series has only finitely many terms, and f must be a polynomial (of degree at most n).

9a Suppose the sequence $f_n(z)$ converges uniformly to $f(z)$ in D. We will show that $f(z)$ is continuous. Let $z_0 \in D$ and suppose $\epsilon > 0$. Choose N so that for every $z \in D$, we have $|f_N(z) - f(z)| < \epsilon/3$. Since f_N is continuous, we can choose $\delta > 0$ so that $|f_N(z) - f_N(z_0)| < \epsilon/3$ whenever $0 < |z - z_0| < \delta$ (moreover, we may assume that δ is small enough so that $D^o(z_0, \delta) \subset D$). Now we know that whenever $z \in D^o(z_0, \delta)$:

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_N(z) + f_N(z) - f_N(z_0) + f_N(z_0) - f(z_0)| \\ &\leq |f(z) - f_N(z)| + |f_N(z_0) - f_N(z)| + |f_N(z_0) - f(z_0)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

so f is continuous, as desired.

9b Let C be a closed contour contained in D. Since the f_n are analytic we have

$$\oint_C f_n(z)dz = 0$$

for every n. We will show that the same is true for the limit f, which will imply by Morera's theorem that f is analytic.

Since the f_n converge uniformly in D they converge uniformly on C as well. Let $\epsilon > 0$ and choose N so that $|f_N(z) - f(z)| < \epsilon$ for all z on C. Then,

$$\left| \oint_C f(z)dz - \oint_C f_N(z)dz \right| \leq \text{length}(C) \cdot \max_{z \in C} |f(z) - f_N(z)| \leq \text{length(C)} \cdot \epsilon.$$

By taking $\epsilon \to 0$ we see that the integral must be zero. Thus,

$$\oint_C f(z)dz = \oint_C f_N(z)dz = 0,$$

and we are done.

10* (a) If $z = 0$ then $f_n(z) = 1$ for all n so $\lim_{n\to\infty} f_n(0) = 1$. If $z \neq 0$ we have

$$\lim_{n\to\infty} |1 + n^2 z| = \infty,$$

so $\lim_{n\to\infty} f_n(z) = 0$.

(b) For any fixed n we have $\lim_{z\to 0} \frac{1}{1+n^2 z} = 1$, since $f_n(z)$ is continuous at 0. Thus,

$$\lim_{n\to\infty} \lim_{z\to 0} f_n(z) = 1.$$

On the other hand, by part (a) we have

$$\lim_{n\to\infty} f_n(z) = 0$$

for every $z \neq 0$, in particular for every z in every punctured neighborhood of 0. Thus,

$$\lim_{z\to 0} \lim_{n\to\infty} f_n(z) = 0,$$

which is different from the first limit.

There are several ways to "explain" why it doesn't converge uniformly, but one way is to appeal to part (a) of question 9: that a limit of a uniformly convergent sequence of continuous functions is always continuous. In this example, the functions $f_n(z)$ are each by themselves continuous in $D(0,1)$, but the limiting function is discontinuous at $z = 0$. Thus, the sequence cannot possibly converge uniformly.

The purpose of this exercise was to demonstrate that you cannot exchange limits in general (whereas you can when you have uniform convergence).