Math 185-5 Fall 2015, Homework 7

Due October 16 in class

1. Brown and Churchill 61.2, 65.3, 72.1, 72.4, 72.8, 72.9, 73.1, 73.6, 73.8.
2. Prove, in a way that is different from the proof in Section 71 of the book, that a power series can be differentiated term by term, i.e., that if
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
convergent in some disk, then
$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1},$$
with the same disk of convergence. (hint: use Taylor's theorem and its converse)
Read sections 71-73 for proofs of the other properties discussed in class.
3. Prove that if $\sum_{n=0}^{\infty} |a_n|$ converges, then
$$\left| \sum_{n=0}^{\infty} a_n \right| \leq \sum_{n=0}^{\infty} |a_n|.$$
4. Calculate the radius of convergence for each of the following power series:
$$\sum_{n=1}^{\infty} (-1)^n z^n / n^3, \qquad \sum_{n=1}^{\infty} z^n / n^n, \qquad \sum_{n=0}^{\infty} n! z^n.$$
Hence determine where they determine an analytic function.
5. Suppose
$$f(z) = \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n.$$
Compute each of the following
$$f^{(6)}(0), \qquad \oint_{|z|=1} \frac{f(z)}{z^4} dz, \qquad \oint_{|z|=1} e^z f(z) dz, \qquad \oint_{|z|=1} \frac{f(z)\sin(z)}{z^2} dz,$$
where all integrals are taken positively.
6. Suppose f is analytic in a domain D and satisfies the functional equation:
$$f(z) = z + f(z^2).$$
Use a Taylor expansion to solve for f.
7. Let $\text{Log}(z)$ be the principal branch of the logarithm and let $\text{Log}_{\pi/2}(z)$ be the branch with $\arg(z) \in (-3\pi/2, \pi/2)$. Find power series expansions of both functions about $z_0 = 2$.

8. Suppose $f(z)$ is entire and there are constants M, n such that $|f(z)| \leq M|z|^n$ for all $z \in \mathbb{C}$. Show that f must be a polynomial.

9. We say that a sequence of functions $\{f_n\}$ converges uniformly to f in a domain D if for every $\epsilon > 0$ there is an N_ϵ such that

$$|f_n(z) - f(z)| < \epsilon \quad \forall z \in D,$$

whenever $n \geq N_\epsilon$.

(a) Rework the argument in Section 70 of the book to show that if the f_n are continuous (but not necessarily analytic) in D then f must be continuous in D. (b) Use Morera's theorem to show that if the f_n are analytic in D then f must be analytic in D. (c) Use Cauchy's integral formula to show that in this case moreover:

$$\lim_{n\to\infty} f_n^{(k)}(z) = f^{(k)}(z),$$

for every k and for all $z \in D$.

10. Cosider the sequence of functions

$$f_n(z) = \frac{1}{1 + n^2 z^2}$$

defined on $D(0, 1)$. (a) Show that

$$\lim_{n\to\infty} f_n(z) = \begin{cases} 1 & z = 0 \\ 0 & z \neq 0 \end{cases}.$$

(b) Compare the values of

$$\lim_{n\to\infty} \lim_{z\to 0} f_n(z)$$

and

$$\lim_{z\to 0} \lim_{n\to\infty} f_n(z).$$

Explain why the sequence f_n does not converge uniformly in $D(0, 1)$.

11. (Optional) Prove that for each $\delta > 0$, the series

$$\zeta(z) := \sum_{n=0}^{\infty} n^{-z}$$

converges uniformly in $\{z : \text{Re}(z) > 1 + \delta\}$. Conclude that $\zeta(z)$ is analytic in $\{z : \text{Re}(z) > 1\}$.

This function is called the *Riemann Zeta Function* and its zeros are closely related to the distribution of prime numbers. The *Riemann Hypothesis* asks whether all these zeros have real part equal to 1/2, and is one of the outstanding unsolved problems in mathematics.