57.4* Let \( h(s) = s^3 + 2s \) and note that \( h \) is entire. When \( z \) is inside \( C \), Cauchy’s integral formula for the second derivative implies:

\[
h''(z) = \frac{2!}{2\pi i} \oint_C \frac{h(s)}{(s-z)^3} \, ds = \frac{g(z)}{i\pi},
\]

so we must have

\[
g(z) = i\pi h''(z) = i\pi (3 \cdot 2 \cdot z + 0) = 6\pi iz,
\]

as suggested.

When \( z \) is outside \( C \), the function \( h(s)/(s-z)^3 \) is analytic on and inside \( C \), since \((s-z)^3 \neq 0\) on and inside \( C \), whence Cauchy-Goursat implies that the integral is zero.

2* Let \( C' \) be a circle centered at \( z \) contained in the interior of \( C \). Since the integrand is analytic in the region between \( C \) and \( C' \), the deformation theorem allows us to replace \( C \) by \( C' \) without changing the integral. So assume \( C \) is a circle centered at \( z \).

Assume \( n \geq 1 \) and assume by induction that we have already proved:

\[
f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^n} \, ds.
\]

Thus, we may write

\[
\lim_{\Delta z \to 0} \frac{f^{(n-1)}(z + \Delta z) - f^{(n-1)}(z)}{\Delta z} = \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}} \, ds - \frac{n \cdot (n-1)!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^n} \, ds.
\]

We now use a trick to avoid messy algebraic calculations — replacing differences by integrals. Observe that

\[
\frac{d}{dw} \frac{1}{(s-w)^n} = \frac{n}{(s-w)^{n+1}}.
\]

Thus, the fundamental theorem of calculus tells us that:

\[
\int_z^{z+\Delta z} \frac{n}{(s-w)^{n+1}} \, dw = \int_z^{z+\Delta z} \frac{d}{dw} \frac{1}{(s-w)^n} \, dw = \frac{1}{(s-z-\Delta z)^n} - \frac{1}{(s-z)^n}.
\]

Even more simply, we may rewrite the second integrand as:

\[
\frac{1}{(s-z)^{n+1}} = \frac{1}{\Delta z} \int_z^{z+\Delta z} \frac{1}{(s-z)^{n+1}} \, dw,
\]
where the integrand is merely a constant with respect to $w$. Substituting these into our limit of interest, we get:

$$\lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint_C f(s) \left( \int_z^{z+\Delta z} \frac{n}{(s-w)^{n+1}} \, dw - \frac{n \cdot (n-1)!}{2\pi i} \oint_C f(s) \frac{1}{\Delta z} \int_z^{z+\Delta z} \frac{1}{(s-z)^{n+1}} \, dw \right) \, ds$$

$$= \lim_{\Delta z \to 0} \frac{n!}{2\pi i \Delta z} \oint_C f(s) \left( \int_z^{z+\Delta z} \frac{1}{(s-w)^{n+1}} \, dw - \int_z^{z+\Delta z} \frac{1}{(s-z)^{n+1}} \, dw \right) \, ds$$

We now apply the same trick again to the integrand with respect to $dw$, namely, observing that

$$\frac{d}{d\zeta} \frac{1}{(s-\zeta)^{n+1}} = \frac{(n+1)}{(s-\zeta)^{n+2}},$$

and therefore

$$\frac{1}{(s-w)^{n+1}} - \frac{1}{(s-z)^{n+1}} = \int_w^w \frac{(n+1)}{(s-\zeta)^{n+2}} \, d\zeta,$$

we can write the above as

$$\lim_{\Delta z \to 0} \frac{n!}{2\pi i \Delta z} \oint_C f(s) \int_z^{z+\Delta z} \int_w^w \frac{n+1}{(s-\zeta)^{n+2}} \, d\zeta \, dw \, ds.$$

We will now show that this limit is zero by applying the ML estimate several times. Let $\max_{s \in C} |f(s)| = M$, and let $m$ be the minimum distance between $z$ and $s \in C$, so $|z-s| \geq m$ for all $s \in C$ (where the max/min exist by continuity and compactness of $C$). Assume without loss of generality that $|\Delta z| < m/2$. Observe that $w \in [z, z + \Delta z]$ and $\zeta \in [z, w]$ and $s \in C$, so in particular we know that $|s-\zeta| \geq m/2$ for all $s \in C$ and $\zeta \in [z, z + \Delta]$. Applying the ML estimate three times, starting from the outermost integral, gives:

$$\left| \oint_C f(s) \int_z^{z+\Delta z} \int_w^w \frac{n+1}{(s-\zeta)^{n+2}} \, d\zeta \, dw \right| \leq \text{length}(C) \max_{s \in C} |f(s)| \max_{s \in C} \left| \int_z^{z+\Delta z} \int_w^w \frac{n+1}{(s-\zeta)^{n+2}} \, d\zeta \, dw \right|$$

$$\leq \text{length}(C) \cdot M \max_{s \in C} \left( \text{length}([z, z + \Delta z]) \max_{w \in [z, z + \Delta z]} \left| \int_w^w \frac{n+1}{(s-\zeta)^{n+2}} \, d\zeta \right| \right)$$

$$\leq \text{length}(C) \cdot M \cdot \max_{s \in C} \left( \text{length}([z, z + \Delta z]) \max_{w \in [z, z + \Delta z]} \text{length}([z, w]) \max_{\zeta \in [z, w]} \left| \frac{n+1}{(s-\zeta)^{n+2}} \right| \right)$$

$$\leq \text{length}(C) \cdot M \cdot |\Delta z| \cdot |\Delta z| \cdot \frac{n+1}{(m/2)^{n+2}}.$$
When \( n \geq 0 \) the integrand is a polynomial, which is analytic on and inside the contour, so by Cauchy-Goursat the integral is zero. Otherwise, for \( n = -k \) where \( k \) is a positive integer we have:

\[
\oint_{|z|=1} z^n(1-z)^m \, dz = \oint_{|z|=1} \frac{(1-z)^m}{(z-0)^k} = \frac{2\pi i}{(k-1)!} f^{(k-1)}(0)
\]

by Cauchy’s integral formula, since \( f(z) = (1-z)^m \) is analytic on and inside \( |z| = 1 \) and 0 is contained in its interior. The derivatives are:

\[
f(0) = 1, \\
f^{(1)}(0) = m(1-0)^{m-1}(-1) = -m, \\
f^{(2)}(0) = m(m-1)(1-0)^{m-2}(-1)^2 = m(m-1), \\
\ldots, f^{(k-1)}(0) = (-1)^{k-1} \frac{m!}{(m-(k-1))!}.
\]

When \( k-1 > m \), i.e. when \( n < -m - 1 \), the derivative is zero since \( f(z) \) is a polynomial of degree \( m \). Thus, the integral is equal to 0 for \( n \geq 0 \) or \( n < -m - 1 \), and

\[
2\pi i(-1)^{-n-1} \frac{m!}{(-n-1)!((m-(-n-1))!} = 2\pi i(-1)^{-n-1} \left( \frac{m}{-n-1} \right),
\]

otherwise.

The formula cannot directly be applied because the function \( \text{Re}(z) \) is not analytic on and inside the contour \( |z| = 1 \), which we will call \( C \). However, since \( \text{Re}(z) = (z+\overline{z})/2 = (z+z^{-1})/2 \) whenever \( |z| = 1 \), the value of the integral (which only depends on the values of \( \text{Re}(z) \) at points on \( C \)) is equal to

\[
\oint_C \frac{1/2(z+z^{-1})}{z-1/2} \, dz = \frac{1}{2} \oint_{|z|=1} \frac{z^2+1}{z(z-1/2)} \, dz.
\]

The integrand has singularities at \( z = 0 \) and \( z = 1/2 \). Let \( C_1 \) and \( C_2 \) be positively oriented circles of radius 1/4 centered at 0 and 1/2 respectively, and observe that since \( C_1 \) and \( C_2 \) are contained in the interior of \( C \) and the integrand is analytic in the region between them and \( C \), we have by the deformation theorem:

\[
\oint_C \frac{z^2+1}{z(z-1/2)} \, dz = \oint_{C_1} \frac{z^2+1}{z(z-1/2)} \, dz + \oint_{C_2} \frac{z^2+1}{z(z-1/2)} \, dz = \oint_{C_1} \frac{g(z)}{z} \, dz + \oint_{C_2} \frac{h(z)}{z} \, dz,
\]

where \( g(z) = \frac{z^2+1}{z-1/2} \) is analytic on and inside \( C_1 \) and \( h(z) = \frac{z^2+1}{z} \) is analytic on and inside \( C_2 \). Thus, by Cauchy’s integral formula, the above integral is equal to

\[
2\pi i g(0) + 2\pi i h(1/2) = 2\pi i(-2 + 5/2) = \pi i,
\]

and the original integral is just \( \pi i/2 \).