Math 185-5 Fall 2015, Homework 6 Selected Solutions

57.4* Let $h(s) = s^3 + 2s$ and note that h is entire. When z is inside C, Cauchy's integral formula for the second derivative implies:

$$h''(z) = \frac{2!}{2\pi i} \oint_C \frac{h(s)}{(s-z)^3} ds = \frac{g(z)}{i\pi},$$

so we must have

$$g(z) = i\pi h''(z) = i\pi(3 \cdot 2 \cdot z + 0) = 6\pi i z,$$

as suggested.

When z is outside C, the function $h(s)/(s-z)^3$ is analytic on and inside C, since $(s-z)^3 \neq 0$ on and inside C, whence Cauchy-Goursat implies that the integral is zero.

2* Let C' be a circle centered at z contained in the interior of C. Since the integrand is analytic in the region between C and C', the deformation theorem allows us to replace C by C' without changing the integral. So assume C is a circle centered at z.

Assume $n \geq 1$ and assume by induction that we have already proved:

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^n} ds.$$

Thus, we may write

$$\begin{aligned}
&\lim_{\Delta z \to 0} \frac{f^{(n-1)}(z+\Delta z) - f^{(n-1)}(z)}{\Delta z} - \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}} ds \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \frac{(n-1)!}{2\pi i} \oint_C \frac{f(s)}{(s-z-\Delta z)^n} - \frac{f(s)}{(s-z)^n} ds - \frac{n \cdot (n-1)!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}} ds.
\end{aligned}$$

We now use a trick to avoid messy algebraic calculations — replacing differences by integrals. Observe that

$$\frac{d}{dw} \frac{1}{(s-w)^n} = \frac{n}{(s-w)^{n+1}}.$$

Thus, the fundamental theorem of calculus tells us that:

$$\int_z^{z+\Delta z} \frac{n}{(s-w)^{n+1}} dw = \int_z^{z+\Delta z} \frac{d}{dw} \frac{1}{(s-w)^n} dw = \frac{1}{(s-z-\Delta z)^n} - \frac{1}{(s-z)^n}.$$

Even more simply, we may rewrite the second integrand as:

$$\frac{1}{(s-z)^{n+1}} = \frac{1}{\Delta z} \int_z^{z+\Delta z} \frac{1}{(s-z)^{n+1}} dw,$$

where the integrand is merely a constant with respect to w. Substituting these into our limit of interest, we get:

$$\lim_{\Delta z \to 0} \frac{1}{\Delta z} \frac{(n-1)!}{2\pi i} \oint_C f(s) \left(\int_z^{z+\Delta z} \frac{n}{(s-w)^{n+1}} dw \right) ds - \frac{n \cdot (n-1)!}{2\pi i} \oint_C f(s) \left(\frac{1}{\Delta z} \int_z^{z+\Delta z} \frac{1}{(s-z)^{n+1}} dw \right) ds$$

$$= \lim_{\Delta z \to 0} \frac{n!}{2\pi i \Delta z} \oint_C f(s) \left(\left(\int_z^{z+\Delta z} \frac{1}{(s-w)^{n+1}} dw \right) - \left(\int_z^{z+\Delta z} \frac{1}{(s-z)^{n+1}} dw \right) \right) ds$$

$$= \lim_{\Delta z \to 0} \frac{n!}{2\pi i \Delta z} \oint_C f(s) \int_z^{z+\Delta z} \left(\frac{1}{(s-w)^{n+1}} - \frac{1}{(s-z)^{n+1}} \right) dw ds$$

We now apply the same trick again to the integrand with respect to dw, namely, observing that

$$\frac{d}{d\zeta} \frac{1}{(s-\zeta)^{n+1}} = \frac{(n+1)}{(s-\zeta)^{n+2}},$$

and therefore

$$\frac{1}{(s-w)^{n+1}} - \frac{1}{(s-z)^{n+1}} = \int_z^w \frac{(n+1)}{(s-\zeta)^{n+2}} d\zeta,$$

we can write the above as

$$\lim_{\Delta z \to 0} \frac{n!}{2\pi i \Delta z} \oint_C f(s) \int_z^{z+\Delta z} \int_z^w \frac{n+1}{(s-\zeta)^{n+2}} d\zeta dw ds.$$

We will now show that this limit is zero by applying the ML estimate several times. Let $\max_{s \in C} |f(s)| = M$, and let m be the minimum distance between z and $s \in C$, so $|z - s| \geq m$ for all $s \in C$ (where the max/min exist by continuity and compactness of C). Assume without loss of generality that $|\Delta z| < m/2$. Observe that[1] $w \in [z, z + \Delta z]$ and $\zeta \in [z, w]$ and $s \in C$, so in particular we know that $|s - \zeta| \geq m/2$ for all $s \in C$ and $\zeta \in [z, z + \Delta]$. Applying the ML estimate three times, starting from the outermost integral, gives:

$$\left| \oint_C f(s) \int_z^{z+\Delta z} \int_z^w \frac{n+1}{(s-\zeta)^{n+2}} d\zeta dw ds \right| \leq \text{length}(C) \max_{s \in C} |f(s)| \max_{s \in C} \left| \int_z^{z+\Delta z} \int_z^w \frac{n+1}{(s-\zeta)^{n+2}} d\zeta dw \right|$$

$$\leq \text{length}(C) \cdot M \max_{s \in C} \left(\text{length}([z, z+\Delta z]) \max_{w \in [z, z+\Delta z]} \left| \int_z^w \frac{n+1}{(s-\zeta)^{n+2}} d\zeta \right| \right)$$

$$\leq \text{length}(C) \cdot M \max_{s \in C} \left(\text{length}([z, z+\Delta z]) \max_{w \in [z, z+\Delta z]} \text{length}([z, w]) \max_{\zeta \in [z, w]} \left| \frac{n+1}{(s-\zeta)^{n+2}} \right| \right)$$

$$\leq \text{length}(C) \cdot M \cdot |\Delta z| \cdot |\Delta z| \cdot \frac{n+1}{(m/2)^{n+2}}.$$

We therefore have

$$\left| \frac{n!}{2\pi i \Delta z} \oint_C f(s) \int_z^{z+\Delta z} \int_z^w \frac{n+1}{(s-\zeta)^{n+2}} d\zeta dw ds \right| \leq \frac{(n+1)! \cdot M \cdot \text{length}(C) |\Delta z|}{2\pi (m/2)^{n+1}},$$

which clearly tends to zero as $\Delta z \to 0$.

6* Let S denote the unit square $\{z = x + iy : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. For a real number x let $\lfloor x \rfloor$ denote the largest integer such that $\lfloor x \rfloor \leq x$. Observe that the hypothesis implies that for every $z = x + iy \in \mathbb{C}$, we have $f(z) = f(z - \lfloor z \rfloor)$ where $\lfloor z \rfloor = \lfloor x \rfloor + i \lfloor y \rfloor$. But since $z - \lfloor z \rfloor \in S$ for every z, this means that $\max_{z \in \mathbb{C}} |f(z)| = \max_{z \in S} |f(z)| \leq M$ for some constant M, since $f(z)$ is continuous on S and S is closed and bounded. Thus f is an entire bounded function, so by Liouville's theorem it must be constant.

[1] We use the notation $[z_1, z_2]$ for z_1, z_2 to denote the line segment connecting z_1 and z_2.

8* When $n \geq 0$ the integrand is a polynomial, which is analytic on and inside the contour, so by Cauchy-Goursat the integral is zero. Otherwise, for $n = -k$ where k is a positive integer we have:

$$\oint_{|z|=1} z^n(1-z)^m dz = \oint_{|z|=1} \frac{(1-z)^m}{(z-0)^k} = \frac{2\pi i}{(k-1)!} f^{(k-1)}(0)$$

by Cauchy's integral formula, since $f(z) = (1-z)^m$ is analytic on and inside $|z| = 1$ and 0 is contained in its interior. The derivatives are:

$$\begin{aligned}
&f(0) = 1, \\
&f^{(1)}(0) = m(1-0)^{m-1}(-1) = -m, \\
&f^{(2)}(0) = m(m-1)(1-0)^{m-2}(-1)^2 = m(m-1), \\
&\ldots, f^{(k-1)}(0) = (-1)^{k-1} \frac{m!}{(m-(k-1))!}.
\end{aligned}$$

When $k - 1 > m$, i.e. when $n < -m - 1$, the derivative is zero since $f(z)$ is a polynomial of degree m. Thus, the integral is equal to 0 for $n \geq 0$ or $n < -m - 1$, and

$$2\pi i(-1)^{-n-1} \frac{m!}{(-n-1)!((m-(-n-1))!} = 2\pi i(-1)^{-n-1} \binom{m}{-n-1},$$

otherwise.

9* The formula cannot directly be applied because the function $\text{Re}(z)$ is not analytic on and inside the contour $|z| = 1$, which we will call C. However, since $\text{Re}(z) = (z+\overline{z})/2 = (z+z^{-1})/2$ whenever $|z| = 1$, the value of the integral (which only depends on the values of $\text{Re}(z)$ at points on C) is equal to

$$\oint_C \frac{(1/2)(z+z^{-1})}{z-\frac{1}{2}} dz = \frac{1}{2} \oint_{|z|=1} \frac{z^2+1}{z(z-\frac{1}{2})} dz.$$

The integrand has singularities at $z = 0$ and $z = 1/2$. Let C_1 and C_2 be positively oriented circles of radius 1/4 centered at 0 and 1/2 respectively, and observe that since C_1 and C_2 are contained in the interior of C and the integrand is analytic in the region between them and C, we have by the deformation theorem:

$$\oint_C \frac{z^2+1}{z(z-\frac{1}{2})} dz = \oint_{C_1} \frac{z^2+1}{z(z-\frac{1}{2})} dz + \oint_{C_2} \frac{z^2+1}{z(z-\frac{1}{2})} dz = \oint_{C_1} \frac{g(z)}{z} dz + \oint_{C_2} \frac{h(z)}{z-\frac{1}{2}} dz,$$

where $g(z) = \frac{z^2+1}{z-1/2}$ is analytic on and inside C_1 and $h(z) = \frac{z^2+1}{z}$ is analytic on and inside C_2. Thus, by Cauchy's integral formula, the above integral is equal to

$$2\pi i g(0) + 2\pi i h(1/2) = 2\pi i(-2 + 5/2) = \pi i,$$

and the original integral is just $\pi i/2$.