Math 185-5 Fall 2015, Homework 6

Due October 9 in class

1. Brown and Churchill 57.2, 57.4, 59.1, 59.8,

2. Prove the Cauchy Integral Formula for higher derivatives. That is, if f is analytic on and inside a simple closed positively oriented contour C and z is interior to C, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}} ds.$$

   Suggested approach: (1) reduce to the case of a circle by applying the deformation theorem. (2) Proceed by induction. Assume you have the formula for $1, \ldots, (n-1)$. Plug this into the definition of the derivative (as a limit of a difference quotient) for $f^{(n)}(z)$. You now need to show that:

$$\lim_{\Delta z \to o} \frac{f^{(n-1)}(z+\Delta z) - f^{(n-1)}(z)}{\Delta z} - \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}} ds = 0.$$

   (3) To avoid messy algebra, use the Fundamental Theorem of Calculus twice:

$$\frac{1}{(s-z-\Delta z)^n} - \frac{1}{(s-z)^n} = \int_z^{z+\Delta z} \frac{n}{(s-w)^{n+1}} dw,$$

   and again as:

$$\frac{1}{(s-z)^{n+1}} = \frac{1}{\Delta z} \int_z^{z+\Delta z} \frac{1}{(s-z)^{n+1}} dw.$$

   Notice that this is where the $n!$ comes from.

   (4) Finally, apply the ML estimate several times.

3. Suppose $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ is a polynomial with $n \geq 1$ and $a_n \neq 0$. Show that there is an R such that

$$|p(z)| > a_n |z|^n / 2$$

   whenever $|z| > R$.

4. Show that if f is entire and $|f(z)| > M$ for all $z \in \mathbb{C}$ then f must be constant.

5. Show that if $f(z)$ is entire and $\mathrm{Re}(f(z)) \leq M$ for all z, then f must be constant.

6. Suppose $f(z)$ is entire and for every z, $f(z) = f(z+1)$ and $f(z) = f(z+i)$. Show that f must be constant.

7. Suppose $f(z)$ is entire and $|f(z)| \leq |z|^{1/2}$ for all z. Show that $f(z)$ must be constant.

8. Evaluate the integral

$$\oint_{|z|=1} z^n (1-z)^m dz \quad m = 0, 1, 2 \ldots \quad n = 0, \pm 1, \pm 2, \ldots,$$

   oriented positively.

9. Explain why Cauchy's integral formula cannot be directly applied to evaluate the integral

$$\oint_{|z|=1} \frac{\mathrm{Re}(z)}{z - \frac{1}{2}} dz,$$

oriented positively.

Then, use the observation that $\mathrm{Re}(z) = \frac{z+z^{-1}}{2}$ whenever $|z| = 1$ to evaluate the integral using CIF.