The questions marked with a * are graded.

2* We have
\[ \frac{1}{z^2 - 1} = \frac{1}{(z - 1)(z + 1)} = \frac{1}{2} \left( \frac{1}{z - 1} - \frac{1}{z + 1} \right), \]
so
\[ \oint_{|z|=2} \frac{1}{z^2 - 1} \, dz = \frac{1}{2} \oint_{|z|=2} \frac{1}{z - 1} - \frac{1}{z + 1} \, dz = \frac{1}{2} \oint_{C_1} \frac{1}{z - 1} - \frac{1}{2} \oint_{C_2} \frac{1}{z + 1} \, dz, \]
where \( C_1, C_2 \) are positively oriented circles of radius \( 1/2 \) centered at \( 1 \) and \( -1 \) respectively, by the deformation theorem. By the fundamental integral, both of the latter integrals are equal to \( 2\pi i \), so the answer is
\[ \frac{2\pi i - 2\pi i}{2} = 0. \]

More concretely, we may write
\[ \oint_{C_1} \frac{1}{z - 1} \, dz = \int_0^{2\pi} \frac{1}{1 + (1/2)e^{it} - 1(1/2)ie^{it}} \, dt = \int_0^{2\pi} i \, dt = 2\pi i, \]
and similarly for \( C_2 \).

3* Suppose \( z \notin C \). and let \( m = \min_{s \in C} |z - s| \), and \( M = \max_{s \in C} |g(s)| \) where the minimum and maximum are achieved because \( C \) is compact and \( |z - s| \) and \( g(s) \) are continuous.

The difference quotient is
\[ \frac{1}{\Delta z} \int_C g(s) \left( \frac{1}{s - z - \Delta z} - \frac{1}{s - z} \right) \, ds = \int_C \frac{g(s)}{(s - z - \Delta z)(s - z)} \, ds. \]

We will show that
\[ \lim_{\Delta z \to 0} \int_C g(s) \left( \frac{1}{(s - z - \Delta z)(s - z)} - \frac{1}{(s - z)^2} \right) = 0, \]
which will complete the proof. Fix \( \epsilon > 0 \). The magnitude of the integrand for \( s \in C \) is at most
\[ |g(s)| \frac{1}{|s - z|} \frac{|\Delta z|}{|s - z - \Delta z||s - z|} \leq M \frac{\Delta}{m^2 (|s - z| - |\Delta z|)} \leq M \frac{\Delta}{m^2 m/2}, \]
whenever \(|\Delta z| \leq m/2\). Thus, by the ML estimate, the integral is at most
\[ \frac{2M\Delta z}{m^3} \cdot \text{length}(C). \]

Since this is less than \( \epsilon \) whenever \(|\Delta z| < \frac{em^3}{2M \cdot \text{length}(C)} \), we are done.
4* There are several ways to do this. One is to observe that the principal branch \( \text{Log}(z) \) is analytic on \( D(1,1) = \{ z : |z - 1| < 1 \} \), so in particular \( \text{Log}(f(z)) \) is analytic on \( D \) since \( f(z) \in D(1,1) \). By the chain rule its derivative is

\[
\frac{1}{f(z)} \cdot f'(z),
\]

so the integrand has an antiderivative in \( D \). Thus the integral is zero for all closed \( C \) contained in \( D \).

Another proof is that \( f(z) \neq 0 \) on \( D \) since \( |f(z) - 1| < 1 \), and \( f'(z) \) is analytic on \( D \) (a consequence of the Cauchy Integral Formula). Thus the quotient \( f'(z)/f(z) \) is analytic on \( D \). Since \( D \) is simply connected, the Cauchy Goursat theorem implies that the integral is zero for every closed \( C \).

Note that the first proof does not use the fact that \( D \) is simply connected.

5* Since \( D \) does not contain the origin \( f(z) = 1/z \) is analytic in \( D \). Since \( D \) is simply connected, the Cauchy-Goursat theorem tells us that \( f \) has an antiderivative \( F(z) \) with \( F'(z) = f(z) = 1/z \) in \( D \). Since \( e^{-z} \) and \( z \) are entire, we may take a composition and a product to conclude that

\[
G(z) = ze^{-F(z)}
\]

is analytic in \( D \). By the chain rule its derivative is

\[
(ze^{-F(z)})' = e^{-F(z)} + z(-e^{-F(z)})F'(z) = e^{-F(z)} + ze^{-F(z)} \cdot (1/z) = 0.
\]

Thus \( ze^{-F(z)} = c \) for some constant \( c \) for all \( z \in D \). Moreover, \( c \neq 0 \) since \( e^{-F(z)} \neq 0 \) and \( z \) is not identically zero in \( D \), which is a nonempty open set. Thus \( e^{F(z)} = c^{-1}z \) in \( D \), and

\[
e^{F(z)+\text{log}(c)} = cz/c = z,
\]

so by the definition of the logarithm as an inverse function, \( F(z) + \text{log}(c) \) (where \( \text{log}(c) \) is any branch whose domain contains \( c \)) is a branch of \( \text{log}(z) \) which is analytic in \( D \).