Math 185-5 Fall 2015, Homework 5 Solutions to Selected Problems

The questions marked with a * are graded.

2* We have

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right),$$

so

$$\oint_{|z|=2} \frac{1}{z^2-1}dz = \frac{1}{2}\oint_{|z|=2} \frac{1}{z-1} - \frac{1}{z+1}dz = \frac{1}{2}\oint_{C_1} \frac{1}{z-1}dz - \frac{1}{2}\oint_{C_2} \frac{1}{z+1}dz,$$

where C_1, C_2 are positively oriented circles of radius $1/2$ centered at 1 and -1 respectively, by the deformation theorem. By the fundamental integral, both of the latter integrals are equal to $2\pi i$, so the answer is

$$\frac{2\pi i - 2\pi i}{2} = 0.$$

More concretely, we may write

$$\oint_{C_1} \frac{1}{z-1}dz = \int_0^{2\pi} \frac{1}{1+(1/2)e^{it}-1}(1/2)ie^{it}dt = \int_0^{2\pi} idt = 2\pi i,$$

and similarly for C_2.

3* Suppose $z \notin C$. and let $m = \min_{s\in C}|z-s|$, and $M = \max_{s\in C}|g(s)|$where the minimum and maximum are achieved because C is compact and $|z-s|$ and $g(s)$ are continuous.

The difference quotient is

$$\frac{1}{\Delta z}\int_C g(s)\left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)ds = \int_C \frac{g(s)}{(s-z-\Delta z)(s-z)}ds.$$

We will show that

$$\lim_{\Delta z\to 0}\int_C g(s)\left(\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2}\right) = 0,$$

which will complete the proof. Fix $\epsilon > 0$. The magnitude of the integrand for $s \in C$ is at most

$$|g(s)|\frac{1}{|s-z|}\frac{|\Delta z|}{|s-z-\Delta z||s-z|} \leq \frac{M}{m^2}\frac{\Delta}{(|s-z|-|\Delta z|)} \leq \frac{M}{m^2}\frac{\Delta z}{m/2},$$

whenever $|\Delta z| \leq m/2$. Thus, by the ML estimate, the integral is at most

$$\frac{2M\Delta z}{m^3}\cdot \text{length(C)}.$$

Since this is less than ϵ whenever $|\Delta z| < \frac{\epsilon m^3}{2M\cdot \text{length}(C)}$, we are done.

4* There are several ways to do this. One is to observe that the principal branch $\mathrm{Log}(z)$ is analytic on $D(1,1) = \{z : |z-1| < 1\}$, so in particular $\mathrm{Log}(f(z))$ is analytic on D since $f(z) \in D(1,1)$. By the chain rule its derivative is

$$\frac{1}{f(z)} \cdot f'(z),$$

so the integrand has an antiderivative in D. Thus the integral is zero for all closed C contained in D.

Another proof is that $f(z) \neq 0$ on D since $|f(z) - 1| < 1$, and $f'(z)$ is analytic on D (a consequence of the Cauchy Integral Formula). Thus the quotient $f'(z)/f(z)$ is analytic on D. Since D is simply connected, the Cauchy Goursat theorem implies that the integral is zero for every closed C.

Note that the first proof does not use the fact that D is simply connected.

5* Since D does not contain the origin $f(z) = 1/z$ is analytic in D. Since D is simply connected, the Cauchy-Goursat theorem tells us that f has an antiderivative $F(z)$ with $F'(z) = f(z) = 1/z$ in D. Since e^{-z} and z are entire, we may take a composition and a product to conclude that

$$G(z) = ze^{-F(z)}$$

is analytic in D. By the chain rule its derivative is

$$(ze^{-F(z)})' = e^{-F(z)} + z(-e^{-F(z)})F'(z) = e^{-F(z)} + ze^{-F(z)} \cdot (1/z) = 0.$$

Thus $ze^{-F(z)} = c$ for some constant c,for all $z \in D$. Moreover, $c \neq 0$ since $e^{-F(z)} \neq 0$ and z is not identically zero in D, which is a nonempty open set. Thus $e^{F(z)} = c^{-1}z$ in D, and

$$e^{F(z)+\log(c)} = cz/c = z,$$

so by the definition of the logarithm as an inverse function, $F(z) + \log(c)$ (where $\log(c)$ is any branch whose domain contains c) is a branch of $\log(z)$ which is analytic in D.