Math 185-5 Fall 2015, Homework 3 Solutions to Selected Problems

The questions marked with a * are graded.

30.6* Let $z = x + iy$. Then

$$|\exp(z^2)| = |\exp(x^2 - y^2 + 2ixy)| = |\exp(x^2 - y^2)||\exp(2ixy)| = \exp(x^2 - y^2).$$

On the other hand, $\exp(|z|^2) = \exp(x^2 + y^2)$. Since $x^2 - y^2 \leq x^2 + y^2$ for real x, y and exp is an increasing function on $\mathbb{R}$, we are done.

36.1* (a) Write $(1+i) = \sqrt{2}\exp(i\pi/4)$. Then,

$$(1+i)^i = \exp(i\log(\sqrt{2}\exp(i\pi 4))) = \exp(i(1/2)ln2 + i(\pi/4 + 2\pi n)).$$

(b)

$$i^{2i} = \exp(2i\log(i)) = \exp(2ii(\pi/2 + 2\pi n)) = \exp(-\pi - 4\pi n),$$

so $1/(i^{2i}) = \exp(-(4n+1)\pi)$.

38.14a*

$$\begin{aligned}\overline{cos(iz)} &= \overline{(\exp(iiz) + \exp(-iiz))}/2 \\ &= (\exp(\overline{i}\overline{i}\overline{z}) + \exp(-\overline{i}\overline{i}\overline{z}))/2 \\ &= (\exp((-i)(-i)\overline{z}) + \exp(-(-i)(-i)\overline{z}))/2 \\ &= (\exp(ii\overline{z}) + \exp(-ii\overline{z}))/2 \\ &= \cos(i\overline{z}).\end{aligned}$$

#2* Let the line be $L = \{a + tb : t \in \mathbb{R}\}$ for some complex $a, b, b \neq 0$. We have then for all $z \in D$, $f(z) = a + tb$ for some $t \in \mathbb{R}$, which implies tht the function $g(z) = (f(z) - a)/b$ is real-valued. But $g(z)$ is also analytic on D since sums and products with constants preserve differentiability, so by a theorem in class g must be constant. But now $f(z) = bg(z) + a$ is constant.

Geometrically, the above calculation amounts to translating and rotating the line L so that it becomes the real line, for which we know how to solve the problem.

#4 Let $z_0 \in D^*$ and let $w_0 = \overline{z_0} \in D$. Since $f'(w_0)$ exists, we have that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{f(w) - f(w_0)}{w - w_0} - f'(w_0)\right| < \epsilon$$

whenever $w \in D^o(w_0, \delta) \subset D$. Notice that each $w \in D^o(w_0, \delta)$ corresponds to a $z = \overline{w} \in D^o(z_0, \delta) \subset D*$ and vice versa (i.e., there is a bijection given by conjugation). Thus, we have that for every $z \in D^o(z_0, \delta) \subset D^*$:

$$\left|\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} - f'(\overline{z_0})\right| < \epsilon,$$

which since conjugation preserves magnitude is the same as

$$\left|\overline{\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}}} - f'(\overline{z_0})\right| = \left|\frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} - \overline{f'(\overline{z_0})}\right| < \epsilon.$$

Thus,

$$\lim_{z\to z_0} \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} = \overline{f'(\overline{z_0})},$$

as desired.

#6* Since $f(z)$ is a branch of log, it is analytic on some domain D and $f(z) = \log(z)$ on D. By the definition this means that $e^{f(z)} = z$ on D. Since both e^z and $f(z)$ are differentiable, we can apply the chain rule, which yields:

$$1 = (e^{f(z)})' = e^{f(z)} f'(z) = z f'(z)$$

on D, whence $f'(z) = 1/z$ on D.

This is not true for z^i. For instance, $\exp(iLog(z))$ and $\exp(i(Log(z) + 2\pi i)) = \exp(-2\pi)\exp(iLog(z))$ differ by a multiplicative constant and thereby have different derivatives.

#7 The function $g(z) = z\exp(\frac{1}{2}\text{Log}(1 - 1/z^2))$ is analytic on $\{z : |z| > 1\}$ because for such z we have $|\text{Re}(1/z^2)| < |1/z^2| < 1$ so $\text{Re}(1 - 1/z^2) > 1 - |\text{Re}(1/z^2)| > 0$ which means $(1 - 1/z^2)$ lies in the domain of analyticity of Log. Moreover,

$$g(z)^2 = z^2 \exp(2 \cdot \frac{1}{2}\text{Log}(1 - 1/z^2)) = z^2(1 - 1/z^2) = (z^2 - 1),$$

so $g(z)$ is a square root of $z^2 - 1$. Thus $g(z)$ is the desired branch.

#8 Let $z \in \mathbb{C} \setminus [0, 1]$. We have

$$\begin{aligned}\lim_{\Delta z\to 0} \frac{1}{\Delta z}\left(\int_0^1 \frac{1}{z + \Delta t - t} dt - \int_0^1 \frac{1}{z - t} dt\right) &= \lim_{\Delta z\to 0} \int_0^1 \frac{1}{\Delta z} \frac{-\Delta z}{(z - t)(z + \Delta z - t)} dt \\ &= \int_0^1 \frac{-1}{(z - t)^2} dt + \lim_{\Delta z\to 0} \int_0^1 \frac{1}{(z - t)^2} - \frac{1}{(z - t)(z + \Delta z - t)} dt\end{aligned}$$

We will show that the latter limit exists and is equal to zero, establishing that $f'(z)$ exists. Let $\epsilon > 0$. Observe that the magnitude of the integrand is:

$$\left|\frac{1}{(z - t)^2} - \frac{1}{(z - t)(z + \Delta z - t)}\right| = \frac{1}{|z - t|} \frac{|\Delta z|}{|z - t||z + \Delta z - t|}.$$

Let $m = \min_{t\in[0,1]} |z - t|$, where the minimum is achieved because $|z - t|$ is continuous and $[0, 1]$ is compact, and moreover $m \neq 0$ because $z \notin [0, 1]$. Thus,

$$\frac{1}{|z - t|} \frac{|\Delta z|}{|z - t||z + \Delta z - t|} \leq \frac{1}{|z - t|} \frac{|\Delta z|}{|z - t|(|z - t| - |\Delta z|)} \leq \frac{|\Delta z|}{m^2(m - |\Delta z|)} \leq \frac{2|\Delta z|}{m^3},$$

where the last inequality holds assuming $|\Delta z| < m/2$. Thus, whenever

$$|\Delta z| < \min\{m/2, m^3\epsilon/2\},$$

we have

$$\left|\int_0^1 \frac{1}{(z - t)^2} - \frac{1}{(z - t)(z + \Delta z - t)} dt\right| \leq \int_0^1 \left|\frac{1}{(z - t)^2} - \frac{1}{(z - t)(z + \Delta z - t)}\right| dt < \int_0^1 \epsilon dt = \epsilon,$$

by the triangle inequality for complex valued functions of a real variable. Thus, the desired limit is zero and $f'(z)$ exists.