Math 185-5 Fall 2015, Homework 3

Due September 18 in class

1. Brown and Churchill 24.4, 24.7, 30.6, 30.7, 33.12, 36.1, 36.6, 38.4a, 38.14.
2. Suppose f is analytic on some domain D, and there is a line $L \subset \mathbb{C}$ such that $f(z) \in L$ for all $z \in D$. Show that f must be constant on D.
3. Suppose f is analytic in a domain D and either $f(z) = 0$ or $f'(z) = 0$ at every point in D. Show that f must be constant in D. (hint: consider f^2).
4. Show that if $f(z)$ is analytic on a domain D, then the function $g(z) = \overline{f(\overline{z})}$ is analytic on the reflected domain $D^* = \{\overline{z} : z \in D\}$, and show that $g'(z) = \overline{f'(\overline{z})}$.
5. Show that $f(z) = \sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$ by using the fact that $f(z)$ is entire and calculating its derivative.
6. Prove that every branch of $\log(z)$ has the same derivative, $1/z$, in its domain. Is this true for every branch of $f(z) = z^i$? (read the section on powers if you attempt this before Monday's class).
7. Find a branch of $f(z) = (z^2 - 1)^{1/2}$ which is analytic in the exterior of the unit disk $\{z : |z| > 1\}$ (hint: write $(z^2 - 1)^{1/2} = z(1 - 1/z^2)^{1/2}$).
8. Show that the function
$$f(z) = \int_0^1 \frac{1}{z - t} dt, \quad z \in \mathbb{C} \setminus [0, 1],$$
is analytic on $\mathbb{C} \setminus [0, 1]$. (hint: compute the derivative by hand.) The integral is just the usual integral of a complex-valued function of the real variable t.
9. (optional) Suppose $u : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives u_x and u_y in a neighborhood of (x_0, y_0). Show that
$$u(x_0 + \Delta x, y_0 + \Delta y) = u(x_0, y_0) + u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \Phi(\Delta x) + \Phi(\Delta y)$$
for some error functions $\Phi(\Delta x) = o(\Delta x)$ and $\Phi(\Delta y) = o(\Delta y)$. (hint: write
$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) = (u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)) + (u(x_0, y_0 + \Delta y) - u(x_0, y_0))$$
Apply the mean value theorem and continuity of the partial derivatives to each part.)
10. (optional) Let $P(x, y) = \sum_{k,\ell=0,\dots N} c_{k,\ell} x^k y^\ell$ be a bivariate polynomial with complex coefficients $c_{k,\ell} \in \mathbb{C}$. Show that
$$P(x, y) = \sum_{j=0,\dots,N} a_j (x + iy)^j$$
i.e., P can be written as a polynomial of the complex variable $z = x + iy$, if and only if
$$\frac{\partial P}{\partial y} = i \frac{\partial P}{\partial x},$$
i.e., P satisfies the Cauchy-Riemann equations.