Math 185-5 Fall 2015, Homework 2 Partial Solutions

The questions marked with a * are graded.

18.9* Choose $\rho > 0$ such that $|g(z)| \le M$ whenever $|z - z_0| < \rho$. Suppose $\epsilon > 0$. Since $\lim_{z\to z_0} f(z) = 0$, we can choose a $\delta' > 0$ such that $|f(z) - 0| = |f(z)| < \epsilon/\rho$ whenever $0 < |z - z_0| < \delta'$. Let $\delta = \min\{\delta', \rho\}$. Then

$$0 < |z - z_0| < \delta \Rightarrow |f(z)| < \frac{\epsilon}{\rho} \quad \text{and} \quad |g(z)| \le M \Rightarrow |f(z)g(z)| = |f(z)||g(z)| < \frac{\epsilon}{\rho} \cdot \rho = \epsilon,$$

so

$$\lim_{z\to z_0} f(z)g(z) = 0,$$

as desired.

24.1bc* **(b)** We have

$$f(x + iy) = z - \overline{z} = 0 + 2iy,$$

so $u_x = u_y = v_x = 0$ and $v_y = 2i$. In particular $u_x \ne v_y$ at all points, so $f(z)$ is not differentiable anywhere.

(c) Write

$$f(x + iy) = e^x(\cos(-y) + i\sin(-y)) = e^x \cos(y) - ie^x \sin(y).$$

Then $u_x = e^x \cos(y)$ and $v_y = -e^x \cos(y)$, so $u_x \ne v_y$ unless

$$e^x \cos(y) = -e^x \cos(y) \iff 2e^x \cos(y) = 0 \iff \cos(y) = 0 \quad \text{since } e^x \ne 0.$$

On the other hand, $u_y = -e^x \sin(y)$ and $v_x = -e^x \sin(y)$, so $u_y \ne -v_x$ unless

$$2e^x \sin(y) = 0 \iff \sin(y) = 0.$$

However, for every y either $\sin(y) \ne 0$ or $\cos(y) \ne 0$ since $\sin^2(y) + \cos^2(y) = 1$. Thus there are no points $z = x + iy$ at which $u_x = v_y$ and $u_y = -v_x$, and f is not differentiable anywhere.

24.2b* We have

$$f(x + iy) = u(x, y) + iv(x, y)$$

for

$$u = (x, y) = e^{-x} \cos y \quad \textit{and} \quad v(x, y) = -e^{-x} \sin y.$$

As these are products of functions that are differentiable everywhere they are differentiable everywhere, with partials:

$$u_x = -e^{-x} \cos y \quad u_y = -e^{-x} \sin y \quad v_x = e^{-x} \sin y \quad v_y = -e^{-x} \cos y.$$

These are continuous everywhere since they are products of continuous functions (exponentials, sine, and cosine), and they satisfy the CR equations. Thus, $f'(z)$ exists and is equal to

$$u_x + iv_x = -e^{-x}(\cos y - i \sin y) = -f(z).$$

Since differentiation is linear, $f''(z) = (-f(z))' = -f'(z) = -(-f(z)) = f(z)$ also exists.

#2 Let ∂S be the set of boundary points of S, and observe from the definition that this is also the set of boundary points of $\mathbb{C} \setminus S$. Then S is open iff $S \cap \partial S = \varnothing$ iff $\partial S \subset \mathbb{C} \setminus S$ iff $\mathbb{C} \setminus S$ is closed.

#3 Suppose S and T are open. Then there are $\epsilon_1, \epsilon_2 > 0$ such that the neighborhoods $D(z, \epsilon_1) \subset S$, $D(z, \epsilon_2) \subset T$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, and observe that

$$D(z, \epsilon) \subseteq D(z, \epsilon_1) \subset S, \quad \textit{and}$$

$$D(z, \epsilon) \subseteq D(z, \epsilon_2) \subset T.$$

Thus, there is a neighborhood of z contained in $S \cap T$ and $S \cap T$ is open. This is no longer true for intersections of infinitely many sets $\bigcap_n S_n$ since the infimum of infinitely many ϵ_n's could be zero. For instance, each of the sets

$$D(0, 1 + 1/n) = \{z : |z| < 1 + 1/n\}$$

is open, but their intersection is the closed disk $\overline{D}(0, 1)$.

For the other part, let $\{S_n\}$ be a (possibly infinite) family of open sets. If $z \in \bigcup_n S_n$ then there is some n for which $z \in S_n$. Since S_n is open there is a neighborhood $D(z, \epsilon) \subset S_n \subset \bigcup_n S_n$, so the union is open.

Remark: I will use $\subset$ and $\subseteq$ interchangeably, both to mean subset of (not necessarily strict).

#4* No. Consider two horseshoe shaped regions which are mirror images of each other, intersecting only at the ends.

#5 See Gamelin pages 11-13.

#6 (a) Let $z_n \to z_0$ be a sequence, and let $\epsilon > 0$ be given. Since $\lim_{z \to z_0} f(z) = w_0$, we can choose a $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$. Choose N so that $|z_n - z_0| < \delta$ for every $n > N$. But now we have $|f(z_n) - w_0| < \epsilon$ for every such n, whence[1] $f(z_n) \to z_0$ as $n \to \infty$.

(b*) We will prove the contrapositive. Suppose $\lim_{z \to z_0} f(z) \neq w_0$. This means that there exists an $\epsilon > 0$ such that each of the punctured neighborhoods

$$D^o(z_0, 1/n) = \{0 < |z - z_0| < 1/n\}$$

where $n = 1, 2, \ldots$ contains a point z_n such that $|f(z_n) - w_0| > \epsilon$. Note that the sequence $z_1, z_2, \ldots$ converges to z_0, since for every $\epsilon' > 0$ every $n > 1/\epsilon'$ satisfies $|z_n - z_0| < \epsilon'$. However the sequence $f(z_1), f(z_2), \ldots$ does NOT converge to w_0, since $|f(z_n) - w_0| > \epsilon$ for all n.

[1] This is a word that I use a lot. In case you haven't heard it before, it means "from which".